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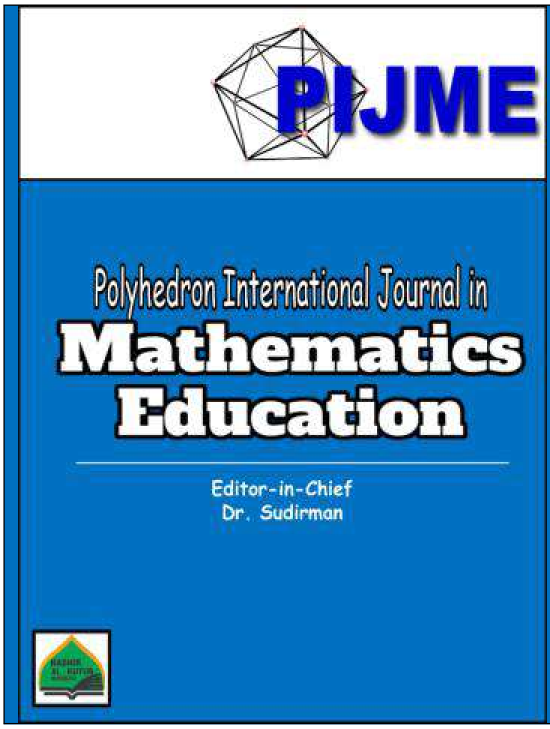

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Original Article

A critical study on the length of Pythagoras' hypotenuse for unit square

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Abstract
This study included the geometric, algebraic, and arithmetic measurement of one-dimensional length for an important ancient theorem in mathematical history, named Pythagoras' theorem, for a unit square. In this paper, we also tried to disclose a rigorous evaluation for the Pythagorean hypotenuses, which were founded incomplete square numbers by us through classical construction instead of considering flexible arithmetic approximation (root extraction) and incommensurable abstract algebraic point of view. Every conscious math reader knows that the characteristic of a theorem is that it produces true results, and the characteristic of a formula is that it may give approximate mathematical results. The much-discussed Pythagorean relation for the side of a right triangle does not satisfy the aspirations of mathematicians to measure the hypotenuse of a unit square, which remains elusive. Therefore, in some cases, this formula exhibits limitations in providing complete results in order to maintain the properties of a theorem.

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1. Introduction

The idea that started about four thousand years ago in the Babylonians, Egyptian, Ancient India (Baudhayana Shulba sutra), China, and at last was institutionalized in the hands of Greek philosopher and mathematician Pythagoras of Samos (570 BC - 495 BC), which we all use as integral tools of the world of mathematics and all branches of science before his name, has not gone out of the question even today (Kalanov, 2013). It means that when another Pythagorean, Hippasus of Metapontum (530 BC - 450 BC), incarnated the concept of irrational numbers, the mathematical world had to deal with a crisis regarding the Pythagoras Theorem because Pythagoras did not believe in irrational numbers. We don't know if he (Pythagoras) did anything to get out of this crisis. But to resolve it, mathematicians tried to maintain the Pythagoras Theorem as arbitrarily applicable to all rational and irrational numbers in various ways, such as the approximation of arithmetic (root extraction), the completeness axiom of real numbers in abstract algebra, which is as good as blind faith in the math world (Kinney, 2019; Lučić, 2015; Math 290, 2014; Wildberger, 2012). But all learned mathematicians admit that the Pythagoras Theorem is quite okay for measuring the area but difficult to accept it strictly for measuring ratio and length (Eriksson, K., Estep, D., Johnson, 2004;(Mansfield, 2023). We think this is uncomfortable for mathematics readers because of the existence of the square root of 2 in the classical construction of numbers in geometry as a length, the existence of such a number in the Real number system whose square is 2 through abstract algebra, and the Pythagoras hypotenuse as the approximate arithmetical value being the square root of 2 are not the same thing. We will try to highlight this point in our study. We are interested in knowing from the readers of this article:

1) We know that algebraic $\sqrt{2}$ is not an axiom; its existence has been proven by us. Now, how did we take an impartial role in establishing the algebraic existence of $\sqrt{2}$ and in proving $(\sqrt{2})^2 = 2$ in algebra?
2) We think we need to prove that $(\sqrt{4})^2 = 4$. So, $(\sqrt{4})^2 = (\sqrt{(2)^2})^2 = (2)^2 = 4$, which is

correct. Now, there is no non-square algebraic x in R such that $(\sqrt{x})^2 = (\sqrt{(y)^2})^2 = (y)^2 = x$, when $y \in R$. So, how can the existence of $\sqrt{2}$ or such algebraic numbers be related to a theorem?

3) What is the correct reasoning regarding the fact that the value obtained from converting a geometric construction into an algebraic number may not return the geometric construction in reverse?
4) Shouldn't only a mathematical statement that yields a 100% true and non-contradictory result be considered a mathematical theorem?

We hope that junior mathematics readers and those interested in working with square root problems will find this article helpful in determining the correct relationship between Pythagoras' theorem and solving square root problems, and they will be able to contribute their valuable thinking regarding this in our mathematical arena.

2. Background

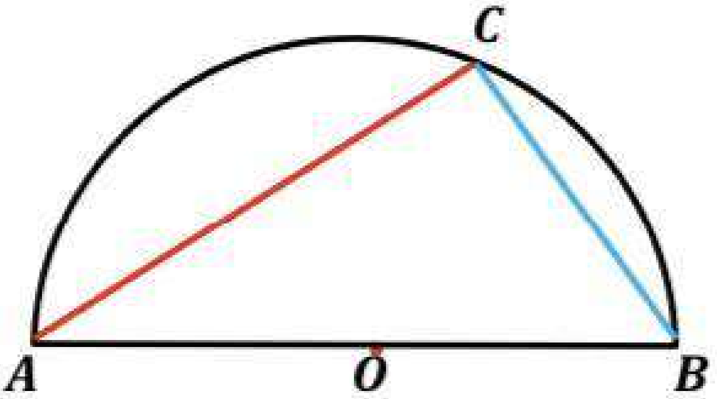
2.1 The two most elegant theorems from Euclidean geometry are described below:

2.1.1 The angle at the circumference of a semicircle is a right angle.

Proof: Suppose $AOBC$ is a semicircle based on the diameter AB whose center is at O. Join A, C, and B, C (ses Figure 1).

Figure 1.

Angle at circumference of semicircle



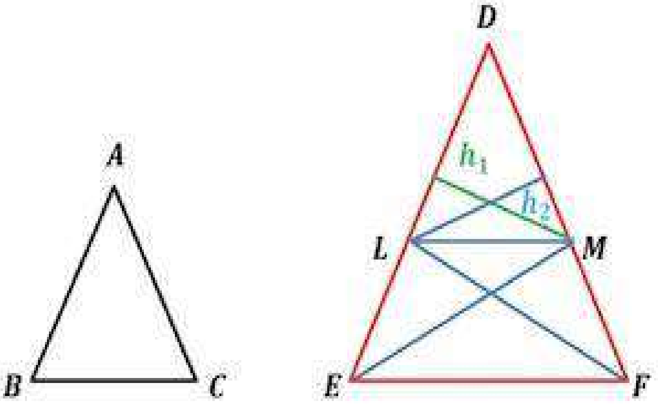
We know that the central angle $\angle AOB = 180°$ is double the angle $\angle ACB$ at the point C on the circumference of that circle. Therefore, $\angle ACB = 90°$.

2.1.2 Fundamental Theorem on similar triangles: If $\Delta ABC \sim \Delta DEF$, then $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

Proof (see Figure 2).

Figure 2

Similar triangles



If $AB = DE$, then the theorem is proved because $\Delta ABC \cong \Delta DEF$, and all the ratios are equal to1.

Suppose, $DE > AB$. We draw the line $LM \parallel EF$ so that $AB = DL$. Since $\angle D = \angle A \therefore DM = AC$.

Now followed by the ratio between the area of the triangles DLM and DEM, we get

$$\frac{\Delta DLM}{\Delta DEM}=\frac{\frac{1}{2}\times DL\times h_1}{\frac{1}{2}\times DE\times h_1}=\frac{DL}{DE}=\frac{AB}{DE}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

Similarly, from the triangles DML and DFL, we get

$$\frac{\Delta DML}{\Delta DFL}=\frac{\frac{1}{2}\times DM\times h_2}{\frac{1}{2}\times DF\times h_2}=\frac{DM}{DF}=\frac{AC}{DF}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Here, $\Delta DFL=\Delta DLM+\Delta MFL=\Delta DLM+\Delta LEM=\Delta DEM$ .... (3)

Therefore, from equations (1), (2), and (3), we have $\frac{AB}{DE}=\frac{AC}{DF}$ .... (4)

With the same process at $\angle F=\angle C$ we can get $\frac{AC}{DF}=\frac{BC}{EF}$ …………….......... (5)

Hence equation (4) and (5) complete the proof $\frac{AB}{DE}=\frac{AC}{DF}=\frac{BC}{EF}$.

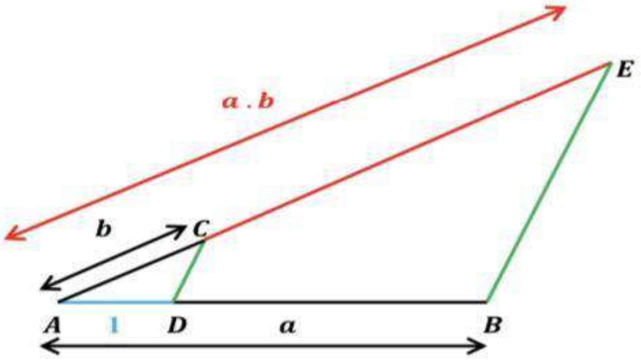
2.2 Description of two classical geometric constructions of numbers here below

2.2.1 Multiplication of two given real numbers

Suppose a and b are two Real numbers whose distances on the Real number line are considered here by the length of the line segments $AB=a$ and $AC=b$. Now we take an acute angle $\angle BAC$ and draw the following geometrical figure in Euclidean plane (see Figure 3).

Figure 3

Construction of Multiplication



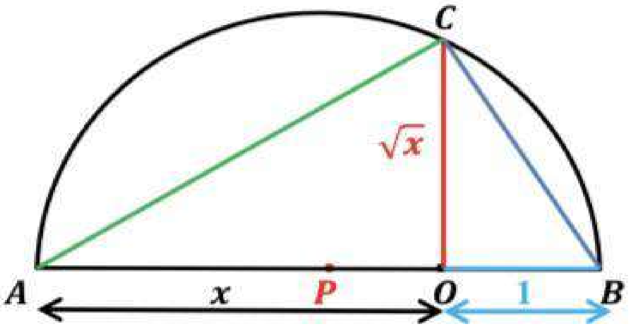
Suppose our straightedge measures the unit length by the line segment AD along the line AB, i.e., $AD=1$. Join D, C. Now draw a line BE parallel to DC, which externally intersects AC at E. From the Fundamental Theorem on similar triangles ADC and ABE, we get $\frac{AE}{AC}=\frac{AB}{AD}$ or $\frac{AE}{b}=\frac{a}{1}$. Therefore, the geometrical construction of the number $a\,.\,b$ is represented here by the length of the line segment AE, i.e., $AE=a\,.\,b$

2.2.2 The square root of a positive Real number

Suppose x is a positive Real number whose distance on the Real number line is denoted here by the length of $OA=x$. We extend the line segment AO by $OB=1$ unit. Draw a perpendicular OC at O to the line AB. Draw a circle considering midpoint P of AB center and $AP=$ radius, which intersect perpendicular OC at C. Join A, C, and B, C (see Figure 4).

Figure 4

Construction of square root



Here, $\angle ACB = 90°$. ΔOAC and ΔOCB are similar whose $\angle ACO = \angle OBC$, $\angle OAC = \angle OCB$. From the Fundamental Theorem on similar triangles OAC and OCB, we get $\frac{OA}{OC} = \frac{OC}{OB}$ or $\frac{x}{OC} = \frac{OC}{1}$. Or, $OC^2 = x \therefore OC = \sqrt{x}$. Therefore, OC represents the length $\sqrt{x}$ geometrically and cannot compute the exact arithmetic value of $\sqrt{x}$ when x is not a perfect square number.

3. Pythagoras' Theorem and its Motive in calculation

3.1 Statement of the Pythagoras Theorem

In a right-angled triangle, the square of the longest side Hypotenuse is equal to the sum of squares of the other two sides named Height and Base, which are inclined to each other at an angle of 90°.

If a right-angled triangle contains two legs of lengths a and b, then the length c of its hypotenuse is related to the formula $a^2 + b^2 = c^2$ and $c = \sqrt{a^2 + b^2}$.

3.1.1 Nature of the Pythagoras Theorem

Pythagoras' Theorem is a mathematical formula that produces an algebraic relationship (sometimes approximate) between the areas of three squares forming on three sides of a right-angled triangle. The Pythagoras Theorem is true in the Euclidean plane with some debates. It does not hold in the hyperbolic or elliptic plane, i.e., in the non-Euclidean geometry. Although it is correct in many cases, its truthfulness is not unquestionable absolutely. Pythagoras' Theorem is not a law in science. It is an algebraic version of the relationship among the area of three squares for a right triangle. Before Hippasus disclosed the existence of irrational numbers, the Pythagoreans believed that all numbers could be expressed as the ratio of two integers, which created a contradiction with themselves (Hurt, 2022). But the theorem statement has remained unchanged for 3000 years in our scientific logic arena. Of course, it is also true that mathematicians are still trying to exhibit their logic and arguments that the theorem is sometimes very mysterious and suspicious, especially in the case of equal legs, incomplete legs, irrational legs, etc., which we will discuss later in our study.

3.1.2 Purposes of the Pythagoras Theorem

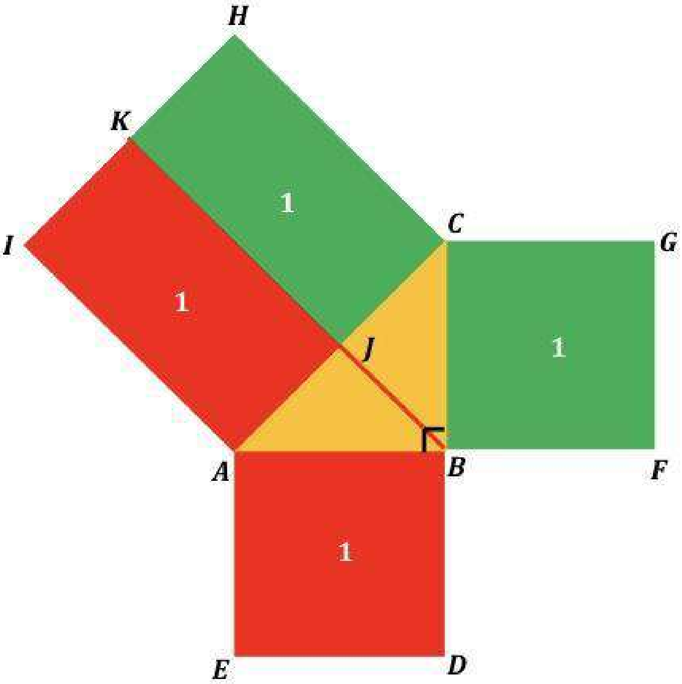
In a single sentence, we can say that the Pythagoras Theorem usually applies in all distance-related branches of science for aiming to arithmetically measure the length of a side of a right-angled triangle when the other two sides are known. It has been spread in all branches of physical science and engineering. Pythagoras' formula is applied to find the length of an object in numbers without directly measuring it in engineering construction and architecture, to determine the shortest distance in navigation, to determine the ground speed of an aircraft for controlling flight, and in many other fields where distance is necessary to measure. For solving a practical life-oriented problem, this formula we cannot use here directly because, through this formula, we calculate the length that has to be solved from a quadratic equation $c^2 = a^2 + b^2$, algebraically. If $a^2 + b^2$ is a perfect square number, then the solution is easy. But, for not complete square, it has become an endless job except approximation, i.e., in a rigorous sense, it does not work accurately.

3.2 Proof of the Pythagoras Theorem

There are at least 371 existing proofs of the Pythagoras Theorem from various corners of mathematics by Scientists and Mathematicians including the father of Geometry Euclid (325 - 265 BC), Astronomer Ptolemy (100 AD – 170 AD), Italian artist Leonardo da Vinci (1452 – 1519), The president of United States James Abram Garfield (1831 – 1881), David Hilbert (1862 – 1943), Great Scientist Albert Einstein (1879 – 1955). Undoubtedly, all such are successful, beautiful, and elegant proofs regarding area measurement.

Euclid gave the proof of the Pythagoras Theorem using only the definitions, postulates, and propositions, and it took a geometric approach followed by area. Euclid proved that the area of the square formed on the longest side can be divided geometrically into two rectangular areas whose areas are the same area formed on the other two sides of the respective right triangle. Based on Euclid's proof, we have taken the figure below for unit square (Kolpas, 2017).

Figure 5
Euclid's proof of Pythagoras Theorem



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In Figure 5, ΔABC is a right-angled triangle with sides $AB = BC = 1$ and hypotenuse $AC = c$. Here, $BJK \perp IH$. According to the Euclid's proof of Pythagoras Theorem, we get, Area $AIKJ =$ Area $ABDE = 1$, Area $JCHK =$ Area $BCGF = 1$. Therefore, Area $ACHI = AIKJ + JCHK = 2 \therefore c^2 = 2$.

3.2.1 The dialectical situation of the Pythagoras Theorem

The Pythagoreans believed that any number can be expressed as the ratio of two whole numbers. They did not believe in irrational numbers. But when Hippasus announced the existence of some numbers beyond Pythagorean thought in our number system while calculating the length of the hypotenuse of a unit square, then mistrust and debate began among mathematicians about the acceptance of the Pythagorean Theorem. The world of mathematics faced a fatal crisis with the number system followed by Pythagoras at that period.

3.2.2 Way out from controversy

Irrational numbers originated from Hippasus' calculation of the square root of two, and the world of mathematics has come a long way since it enriched mathematics. In construction and engineering, they advanced using the Pythagoras Theorem by extracting the square root of positive real numbers as approximate arithmetic values that are not perfect squares and recognizing the Pythagoras Theorem. During the transition of acceptance of Pythagoras' theorem, many mathematicians proved that $(\sqrt{2})^2 = 2$ and subsequently established that $\sqrt{2}$ is an irrational number recognized as the Pythagorean hypotenuse. On the other hand, Algebraists argued abstractly for the expansion of real numbers, placing $\sqrt{2}$ in the Real number system and saying that since $(\sqrt{2})^2$ is equal to the area of a square based on the diagonal of a unit square, the theorem is correct. We also want to agree here that $(\sqrt{2})^2 = 2$, but before that, we need to find out the value of the square root of two in an exact form. We consider this to be an endless process. Although it is possible to express $\sqrt{2}$ from the classical construction in geometry by the length of a line segment, it is impossible by an exact arithmetic amount. If we truncate the value of the square root of two at certain decimal places, it will strictly be less than the actual value, and if we round off at a desired decimal place, it will have the chance to be greater than the actual value. However, in the next section, we will check the geometrical construction validity of the length of the hypotenuse of the unit square as the square root of two through the straightedge-compass principle and whether the algebraic measurement length of the hypotenuse of a unit square should be $\sqrt{2}$, exactly.

4. Three debatable notions of the existence of the square root of two

Pythagoras did not introduce his theorem to establish the existence of $\sqrt{2}$ as an irrational number as if its square is 2 because irrationality did not belong to Pythagoras' thinking. We, the followers of Pythagoras, have been trying to establish the existence of $\sqrt{2}$ by aiming to re-establish that the Pythagorean Theorem was for constructing irrational $\sqrt{2}$ in his described hypotenuse. We want to say that the respective theorem was not prescribed there before our math realm after resolving the irrationality issue for $\sqrt{2}$. We are also analogously trying to mitigate the problem of holding the Pythagoras Theorem by adopting an inexact number symbolized with $\sqrt{2}$ as an exact measurement of the hypotenuse, a painful discomfort for math readers. However, we think of three concepts that usually work in us concerning the existence of $\sqrt{2}$, which are as follows.

1) The square root of two is hidden forever in the compass length

Following subsubsection 2.2.2 of the classical geometric process, setting $x = 2$ yields a line segment corresponding to the square root of two whose geometric length exists in compass, just like the square root of a perfect square number and its square $= 2$. Such geometric length constructions are truth figures against numbers taken similar to the length of the closed circular path as the number $2\pi r$. It also intersects the Real number line at a point but, numerically the exact value of that intersecting point cannot be retrieved as because it is a mathematical process that is forever unfinished. However, at the compass measurement, a geometric length of $\sqrt{2}$ is absolute and unique, i.e., its extent is definite, limited, and unchangeable. Among the three understandings of the square root of two, this is the noblest and most logically complete because there is no possibility and no compulsion to make the object in its shape less or more by the compass. Therefore, the square root of two as the compass length undoubtedly exists and is measurable geometrically.

2) The square root of two is a theoretical unwritten value forever in abstract algebra

According to the definition of square root, if the square root of x is y, then $y^2 = x$. This argument works here without finding the exact or finished value of $\sqrt{2}$. Concerning that argument, it has been well-established in our geometry section that the square root of two exists abstractly, where $(\sqrt{2})^2 = 2$. But when asked to find the value of $\sqrt{x}$, none of us can ever find $y = \sqrt{x}$ unless x is a perfect square number, i.e., here, we do not have the opportunity to write the exact physical value of the square root of two.

Suppose we have two machineries $g(x) = x^2$ and $f(x) = \sqrt{x}$. A raw material here $x = 4$ produces a specific product under composite activities $(gof)(x)$, where the product is $g(f(x)) = g(\sqrt{4}) = g(2) = (2)^2 = 4$, and it does not follow the intuitive or indirect method. Now, for another raw material $x = 2$, we fall in an infinite loop because machinery $f(x) = \sqrt{x}$ cannot yield a specific product unless we manipulate the production procedure of $f(x)$ as only a symbol $\sqrt{x}$. In such cases, the respective activities $(gof)(x)$ can complete their procedure if the system goes under an arithmetic approximation with imperfection. So, in some cases, the activities $(gof)(x)$ properly do not work. Rigorously, for more clearance, $\sqrt{2}$ is not a number but rather an approximate arithmetical operation, a notion, or a symbol only. We should not avoid the reality of incompleteness of $\sqrt{2}$ for aspects of mathematical rules, i.e., the characteristics of a statement to be a theorem, are intimately involved. An interesting thing is that there will come a time when $y = a$, a complete numerical value, and $a^2 = 2$ will occur. We are blind imitators if we can't tell when we will be lucky to see such a moment. In this case, without seeing the complete form of the square root of two, we accept its existence and that its square is two. So, the exact algebraic value of the square root of two remains abstract in calculating square root history like an endless path.

3) The square root of two is a practical arithmetical value at any decimal place in approximation

The Pythagoras Theorem has said from his empirical knowledge that the diagonal length of a unit square has a specific arithmetic value c so that $c^2 = 2$, practically. According to the concepts mentioned in (1) and (2), we get no specific practical arithmetic value of $\sqrt{2}$. Per contra, we get an incomplete algebraic value of $\sqrt{2}$ from square root extraction. So, there is no substitute except approximation to get a practical arithmetic value. For example, we can get approximate arithmetic values, such as $c = 1.42$ (rounded off at the upper), $c = 1.4142, 1.41$ (truncated),…., etc. These re-processing values are larger or smaller than the actual value of $\sqrt{2}$ and no procedure can write the exact value of $\sqrt{2}$. It is impossible to create the exact practical arithmetical value of the actual length of the hypotenuse $\sqrt{2}$ from the calculation of Pythagoras' formula. The actual findings is that here, $\sqrt{2}$ behaves like a variable rather than a constant like gravitational acceleration g , i.e., $\sqrt{2} = 1.4142, 1.42, 1.41, \dots$ are all fine. Therefore, according to the consequence of calculation, the Pythagoras formula can produce values closer to the actual length of the hypotenuse, which is not useless to accept but is crippling to strict mathematical rules. The world's famous Pythagorean Theorem was disclosed before the invention of irrational numbers when the Pythagorean hypotenuse assumed rational numbers. Despite irrationals being dignified by the Real number system, it is no longer revised to establish Pythagoras' theorem. Besides, it recognized the rationalization of irrational numbers in hypotenuse measures. In this approximation, we may work with separate approximations of such numbers in different places for different needs. But, if a theorem produces such indeterminate results, there is a chance to lose its validity. The Pythagorean Theorem is correct, followed by the approximation process. So, this world-famous theorem of not being able to find the exact square root of a number that is not a square exists. In this case, it deserves analyzing how logical it is to consider it the theorem.

Comment

Among these three concepts, it is clear that the accurate length of $\sqrt{2}$ can be realized in the geometric process through the compass as a line segment or a geometrical object. Hence, the existence of the square root of two is appropriate in the classical construction as only length shape object, not in digit. Other processes are unspecified or symbol-dependent only.

5. Main Results

5.1 Measurement of $\sqrt{a^2 + b^2}$ through Pythagorean hypotenuse and classical geometrical construction

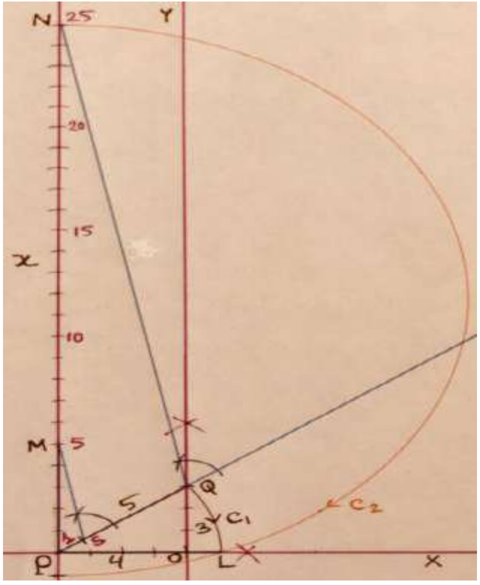
Mathematically, we all know that the Pythagoras Theorem expresses the algebraic relation $a^2 + b^2 = c^2$, where if any two of a, b, c are known, then the rest will be known to us for usual notations. Although the theorem describes the relationship between areas, it mainly measures length in practical terms. The value we obtain aiming for the length of the hypotenuse by the theoretical formula of the well-known Pythagoras Theorem does not always rigorously correspond to the original geometric length of the hypotenuse because, for particular cases when the respective theoretical formula is forced there for arithmetic approximation, the value measured by it does not strictly represent the truth geometric length of the hypotenuse. The theoretical value is slightly more or less than the geometric length of the hypotenuse. In the following geometrical constructions, therefore, when we choose the geometrical length of the hypotenuse of a right triangle using the Pythagorean theoretical standard, it must be increased or decreased by a microscopic amount based on the approximation of its original length in compass measurement. The reason for taking this is that for a statement to be a theorem, it must be 100 percent true. It is necessary to verify whether the approximations can exist in the theorem. So, here, we will see whether the geometrical construction is possible from the arithmetic approximations of the hypotenuse and what deviation might be created in such an approximation.

In this section, we will try to verify whether the approximate arithmetic length value of the Pythagoras hypotenuse satisfies the straightedge and compass construction of square root in Euclid geometry. We will also try to test under the same principle whether multiplying the approximate arithmetic length of the Pythagoras hypotenuse with itself can regain the original scale unit along the number line.

5.1.1 Verification of the length c of the hypotenuse and its square c^2 when $a^2 + b^2$ is a perfect square number

If we choose $\boldsymbol{a} = \mathbf{4}$ and $\boldsymbol{b} = \mathbf{3}$, then $a^2 + b^2 = 16 + 9 = 25$ is a perfect square number. Now we construct the following geometric objects aiming to achieve the geometric length of the square root of a perfect square number followed by sub-subsections 2.2.1 and 2.2.2 and length against Pythagoras hypotenuse (see Figure 6).

Figure 6
Verification of Pythagoras Theorem for $4^2 + 3^2 = 5^2$



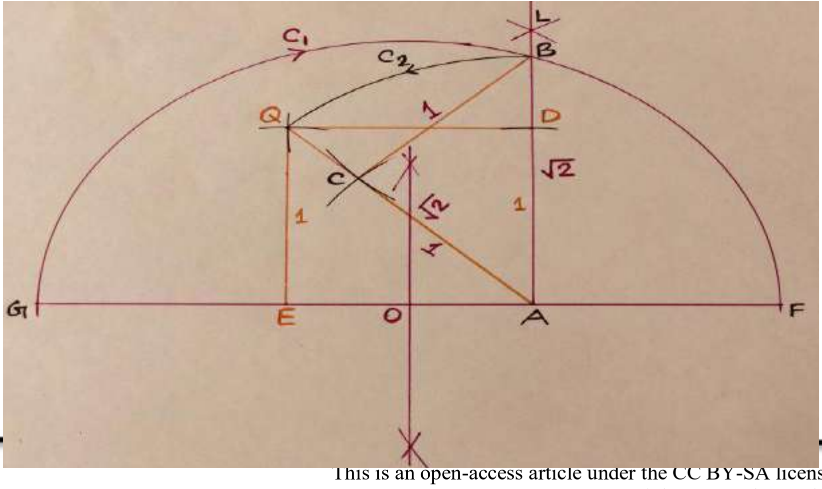
In Figure 6, $PX \perp PN, PX \perp LY$, Pythagorean legs are $PO = 4, OQ = 3$. Let the Pythagoras hypotenuse is $PQ = c$. Here, $PL = PM = 5, PN = 25$. Draw a circle C_1 centering P and radius $PQ = c$. Circle C_1 passes through points L and M, which proves the length of the Pythagoras hypotenuse is $PQ = c = 5$. Again, from the classical construction of $\sqrt{25}$, we see that Circle C_2 passes through point L, which proves the length of the Pythagoras hypotenuse is 5. Now let $\sqrt{25} \times \sqrt{25} = x$. From the straightedge-compass construction, we get $x = PN = 25$, which means the multiplication of the length of the Pythagoras hypotenuse with itself regains the original scale. So, in this case of $a^2 + b^2$ being a perfect square number, the truthfulness of the Pythagoras Theorem is verified successfully.

Similarly, we can verify the fact for other triples as $(0.3, 0.4, 0.5)$, $(5, 12, 13)$, $(6, 8, 10)$, $(0.6, 0.8, 1)$, $(9, 12, 15)$, $(15, 20, 25)$, $(0.15, 0.2, 0.25)$, ……, etc., i.e., the Pythagoras Theorem is correct for such combinations (a, b, c), where the term $(a^2 + b^2)$ is always a perfect square number.

5.1.2 Geometrical Construction of $\sqrt{2}$ and existence of $\sqrt{2}$ as a geometric length or object in a right triangle through straightedge-compass

We consider the figure below.

Figure 7
Geometrical construction for $\sqrt{\mathbf{2}}$



In Figure 7, arbitrarily taken $\boldsymbol{AF=AE=GE=AD=DQ=QE=1}$, $AL \perp GF$, and O is the midpoint of GF. Draw a circle C_1 with the center of O and a radius of $OG = 1.5$, which intersects AL at point B. So, according to sub-subsection 2.2.2, AB represents the length $\sqrt{2}$ geometrically. Now draw two circles centering A and B with unit radii, which intersect each other at point C on the diagonal AQ of the unit square $ADQE$. Also, circle C_2 with the center of A and a radius of hypotenuse $= AB = \sqrt{2}$ fulfills the diagonal or hypotenuse length $AQ = \sqrt{2}$.

Here, we can see that the number $\sqrt{2}$ can be classically constructed and shown as the length of the hypotenuse of a unity-sided right triangle. So, it seems that the Pythagorean Theorem is 100 percent true. But is it possible to accept without hesitation? According to the Pythagorean Theorem, after determining the approximate arithmetic position of $\sqrt{2}$ on the number line, if we return to the geometric construction from it, the properties of the theorem do not hold. It destroys the purity of the geometric construction. We can start with an example. Suppose we need to know which number between $3\sqrt{2}$ and $2\sqrt{5}$ is greater concerning arithmetic (root extraction) value. These exist in geometric lengths, but we cannot easily guess which is the longer. In this case, we must come out of the two symbolic expressions of $3\sqrt{2}$ and $2\sqrt{5}$ to form 4.243 and 4.472 respectively greater and smaller than their original value. If we want to return to building their original geometric lengths with the approximate values 4.243 and 4.472, we will get the error because the originals are shorter and longer, respectively. Furthermore, in this context, we think that in engineering construction, we have to make a ramp and pay its construction cost where the length of the ramp is $\sqrt{2}$ units. If construction workers do not provide a ramp length of at least 1.42 units, it will fail to create a safe structure. Again, we consider the ramp length of 1.41 units while paying the construction cost. If we take the length of the ramp as correct in terms of the physical structure of the building, the percentage of error in the construction cost is 0.704. Again, if we take the standard adopted in the case of construction cost as correct, then the percentage rate is 0.709. It is our practical situational obligation, and there is no reason to go beyond such, i.e., we are helpless in all these cases except approximations. So, can we say that no mathematical theorem holds if it accepts approximation when the statement does not give a 100 percent truth result? In the following subsections, we will try to express how deviations in geometric classical constructions will have to occur when we finish our work with such approximations in the case of Pythagorean hypotenuse.

5.2 Verification of the length c of the hypotenuse and its square c^2 when $a^2 + b^2$ is not a perfect square number

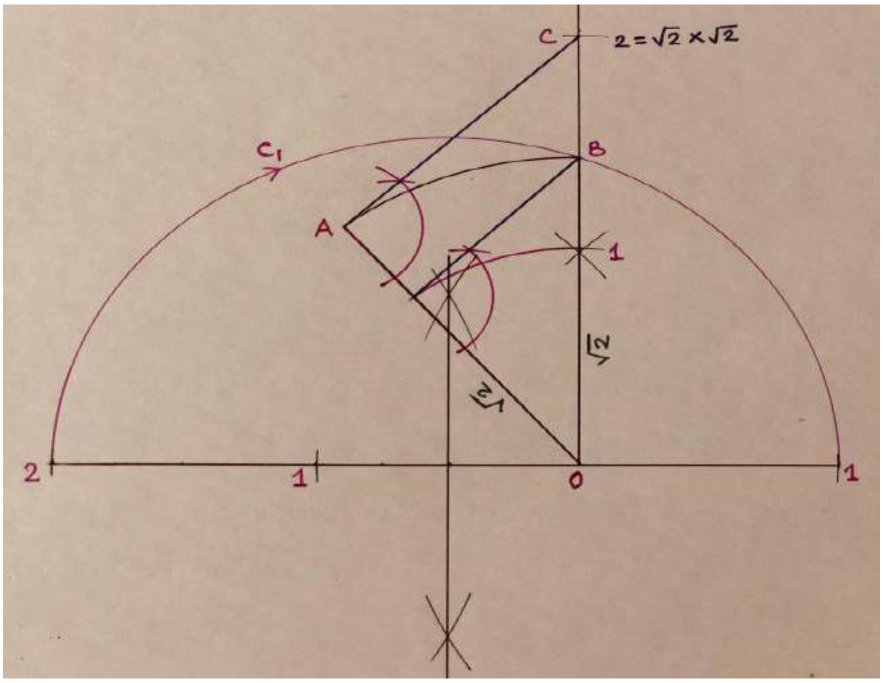
Pythagoras' formula can measure the hypotenuse length in complete form when $\boldsymbol{a^2+b^2}$ is a perfect square number. Otherwise, it needs to take an arithmetic approximation scheme on $\sqrt{\boldsymbol{a^2+b^2}}$ to mitigate the controversy (Steihaug, 2024) of Pythagorean philosophy about the number system because none of the numbers that are not square of a rational number is a perfect square number. In that case, the formula of the Pythagoras Theorem cannot measure it with an exact form. In the geometric classical construction of numbers, we can form a truth length against the square root of each rational number. There is no need to take length approximation in classical construction. But, to regain the geometric length shape for hypotenuse from the algebraic approach of Pythagoras' formula, we must take an arithmetic approximation on $\sqrt{\boldsymbol{a^2+b^2}}$. In the following sub-subsections, we are trying to study the facts of regaining truth geometrical length or object construction produced from the compulsory approximate arithmetic value of the Pythagorean algebraic hypotenuse.

5.2.1 Classical geometric construction of $\sqrt{2}$ and $\sqrt{2} \times \sqrt{2}$

There are a lot of proofs by contradiction for the existence of $\boldsymbol{x}$ in the set of Real numbers such that $\boldsymbol{x^2 = 2}$, evidently which is related to $\sqrt{\boldsymbol{2}}$ is an irrational number (Math 290, 2014; Wildberger, 2012; Eriksson, K., Estep, D., Johnson, 2004; Shah, 2021; Gasarch, William & Kruskal, Alexander & Kruskal, Justin & Kruskal, 2006; Cambridge, n.d.). We are here trying to prove the existence of such $\boldsymbol{x}$

that $\boldsymbol{x^2 = 2}$ from the geometrical point of view because we believe that $\sqrt{\mathbf{2}}$ cannot be uttered arithmetically in a complete sense. To complete the task, we can construct it in the following drawing.

Figure 8
Geometrical construction of $\sqrt{\mathbf{2}}$ *and* $\sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}}$



In Figure 8, $\boldsymbol{OB} = \sqrt{\mathbf{2}}$ has been constructed here following the circle $\boldsymbol{C_1}$ and sub-**s**ubsection 2.2.2. Again, considering acute angle $< BOA$ between $\sqrt{\mathbf{2}}$ representing two line segments $\boldsymbol{OB} = \boldsymbol{OA} = \sqrt{\mathbf{2}}$, we here construct the multiplication $\sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}}$ followed by sub-subsections 2.2.1**,** which retrieves the original scale $\boldsymbol{OC} = \mathbf{2}$. Therefore, $\boldsymbol{OC} = \sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}} = \mathbf{2}$ simultaneously proves the existence of $\sqrt{\mathbf{2}}$ and its square, i.e., $(\sqrt{\mathbf{2}})^{\mathbf{2}} = \mathbf{2}$ in geometrical constructions.

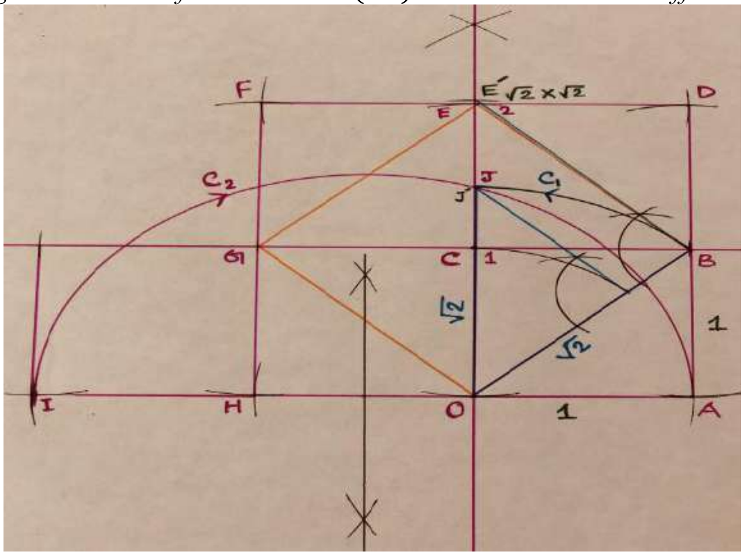
In sub-subsection 5.1.1, we observed that the hypotenuse obtained from the Pythagoras algebraic formula $\boldsymbol{c} = \mathbf{5}$ fits the classical geometric construction. Now, we will see whether similar geometric constructions can be regained from their respective algebraic value of Pythagorean hypotenuse. For example, suppose we get the value $\sqrt{\mathbf{111}}$ for a hypotenuse from Pythagoras' algebraic formula $\boldsymbol{c^2} = \boldsymbol{a^2} + \boldsymbol{b^2}$. The respective produced non-specific symbolic number $\sqrt{\mathbf{111}}$ can never be attributed to verify its compatibility with the classical geometric construction because it does not represent an understandable arithmetic amount in a complete sense. In this case, we must verify by taking its approximate values, e.g., $\mathbf{10.54}$ (rounded off at the upper), like $\sqrt{\mathbf{111}} + \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} > \mathbf{0}$. We are trying to examine this in the next sub-subsections. We will do this to test whether the Pythagorean Theorem statement is 100 percent true.

5.2.2 Verification of the length of the hypotenuse and its square for a unit square after an arithmetical approximation of $\sqrt{2}$ by rounding off at the upper

In sub-subsections 5.1.2 and 5.2.1, we mentioned that we must take the approximate next decimal places value of $\sqrt{\mathbf{2}}$ at the minimum level since we want to solve practical life-oriented problems in our distance-related measurements by applying the Pythagoras formula. With the similar process described in the sub-subsections 5.1.1 and 5.2.1, we complete the construction below.

Figure 9

Verification of Pythagoras Theorem for $1^2+1^2=(\sqrt{\mathbf{2}})^2$ *when* $\sqrt{\mathbf{2}}$ *rounded off*



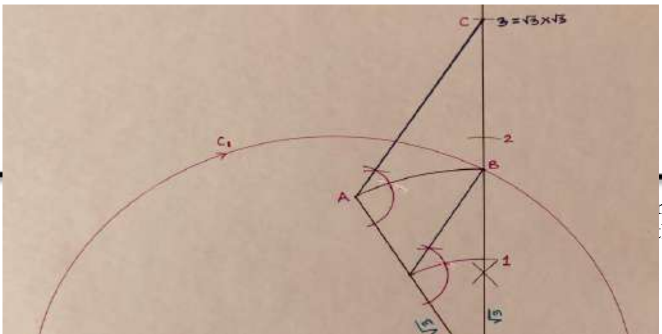
In Figure 9, $OA \perp OC, OC \parallel AD \parallel HF, OA = OH = HI = OC = CE = 1$. Since $OABC$ is a unit square, the Pythagoras Theorem produces $OB = \sqrt{2}$. According to the Pythagorean calculation, we take such an arithmetic approximation of $\sqrt{2}$, which we can geometrically fix at point $(OB + \varepsilon)$, where ε is a small positive quantity that can be estimated at the compass while drawing. It is noteworthy that such an approximation has to be taken in the case of the formula of the Pythagorean Theorem. Draw a circle C_1 centering O and radius $(OB+\varepsilon)$. Circle C_1 intersects the line OE at point J ∴ Pythagoras' approximation says that $OJ = \sqrt{2}$. But, while a circle C_2 is drawn, under the classical construction of $\sqrt{2}$, it does not pass through the point J, although it meets at I and A. It is seen that circle C_2 passes through a point J' on the line OE very near point J, where $OJ' < OJ$, and looks like microscopically $OJ > OJ'$. Again, if we construct $\sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}}$ based on the Pythagoras approximate hypotenuse $\boldsymbol{OJ} = \sqrt{\mathbf{2}}$ using straightedge-compass, it produces a length $\boldsymbol{OE}'$, which is longer than the original geometric length $\boldsymbol{OE} = \mathbf{2}$, i.e., $\boldsymbol{OE}' > OE$. So, from the arithmetic approximation, the original geometric length could not be regained here.

Since $\boldsymbol{OJ}' = \sqrt{\mathbf{2}}$ and $\boldsymbol{OE} = \sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}} = \mathbf{2}$ have been coming from the truth straightedge and compass construction, and a subtle error for length shape is observed here with the formula of the Pythagoras arithmetic approximation, the phenomenon does not give us scope to accept the Pythagorean approximate arithmetic measurement of $\boldsymbol{OJ} = \sqrt{\mathbf{2}}$ and $\boldsymbol{OE}' = \sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}} = \mathbf{2}$ as a geometric entity. We have the following construction by adopting the same procedure to check the result obtained in the sub-subsections 5.2.1 and 5.2.2, also hold for $\sqrt{\mathbf{3}}$.

5.2.3 Classical construction of $\sqrt{\mathbf{3}}$ and $\sqrt{\mathbf{3}} \times \sqrt{\mathbf{3}}$

Figure 10

Geometrical construction of $\sqrt{\mathbf{3}}$ *and* $\sqrt{\mathbf{3}} \times \sqrt{\mathbf{3}}$



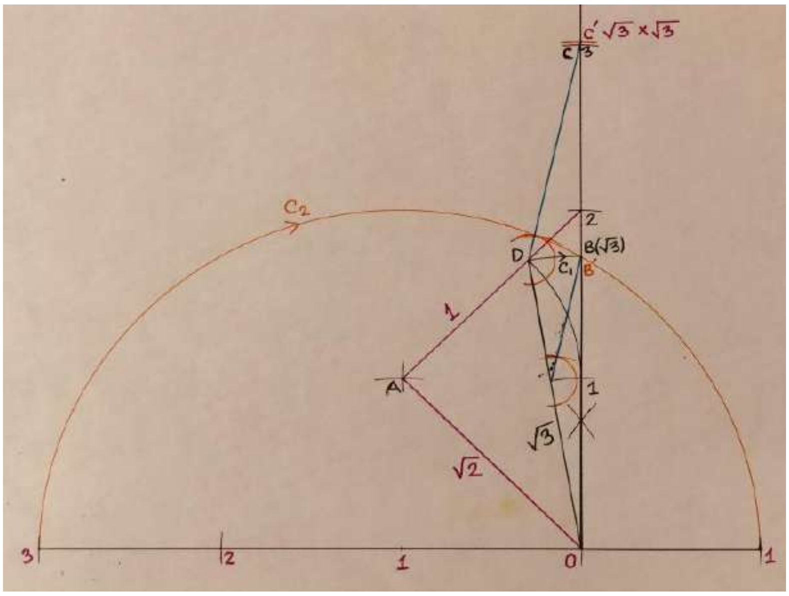
In Figure 10, $\boldsymbol{OB} = \sqrt{\mathbf{3}}$ has been constructed here following the circle $\boldsymbol{C_1}$ and sub-subsection 2.2.2. Again, considering acute angle $< BOA$ between $\sqrt{\mathbf{3}}$ representing two line segments $\boldsymbol{OB} = \boldsymbol{OA} = \sqrt{\mathbf{3}}$, we here construct the multiplication $\sqrt{\mathbf{3}} \times \sqrt{\mathbf{3}}$ followed by sub-subsections 2.2.1**,** which retrieves the original scale $\boldsymbol{OC} = \mathbf{3}$. Therefore, $\boldsymbol{OC} = \sqrt{\mathbf{3}} \times \sqrt{\mathbf{3}} = \mathbf{3}$ simultaneously proves the existence of $\sqrt{\mathbf{3}}$ and its square, i.e., $(\sqrt{\mathbf{3}})^{\mathbf{2}} = \mathbf{3}$ in geometrical constructions.

5.2.4 Verification of the length $\sqrt{\mathbf{3}}$ of the hypotenuse and its square $(\sqrt{\mathbf{3}})^{\mathbf{2}}$ after an arithmetical approximation of $\sqrt{\mathbf{3}}$ by rounding off at the upper in sequence of $\sqrt{\mathbf{2}}$

We consider the figure below.

Figure 11

Verification of Pythagoras Theorem for $(\sqrt{\mathbf{2}})^{\mathbf{2}} + 1^2 = (\sqrt{\mathbf{3}})^{\mathbf{2}}$ *when* $\sqrt{\mathbf{3}}$ *rounded off*



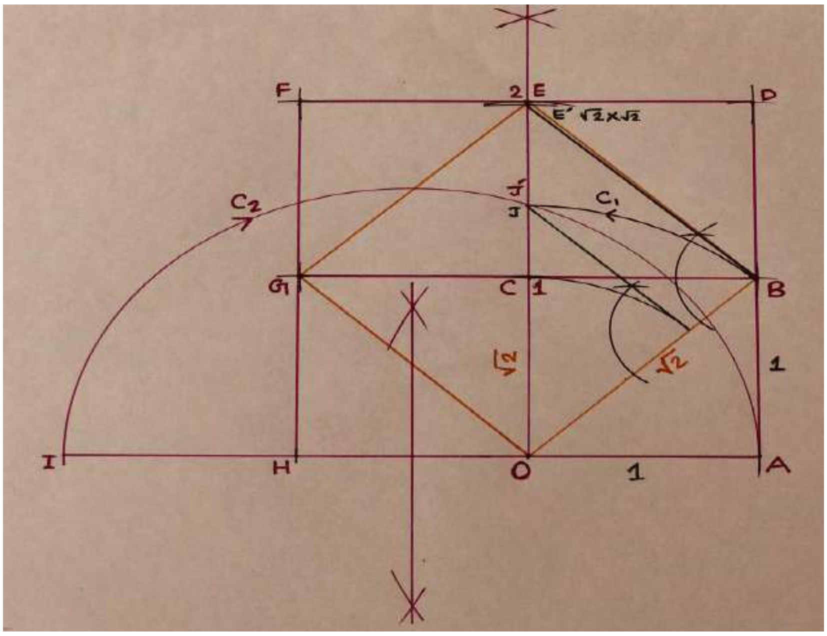
In Figure 11, $\angle OAD = 90°$, $AD = 1$ $\therefore$ the Pythagoras Theorem produces $OA = \sqrt{2}$, $OD = \sqrt{3}$. Now, we approximate $\sqrt{3}$ arithmetically as before in 5.2.2 at $(OD + \varepsilon)$. Draw a circle C_1 centering O and radius $(OD + \varepsilon)$. Circle C_1 intersects the line OC at point B $\therefore$ Pythagoras' approximation says that $OB = \sqrt{3}$. But, while a circle C_2 is drawn, under the classical construction of $\sqrt{3}$, it does not pass through the point B. It is seen that circle C_2 passes through a point B' on the line OC very near point B, where $OB' < OB$, and looks like microscopically $OB > OB'$. Again, if we construct $\sqrt{\mathbf{3}} \times \sqrt{\mathbf{3}}$ based on the Pythagoras approximate hypotenuse $\boldsymbol{OB} = \sqrt{\mathbf{3}}$ using straightedge-compass, it produces a length $\boldsymbol{OC'}$**,** which is longer than the geometric original length $\boldsymbol{OC} = \mathbf{3}$**,** i.e., $\boldsymbol{OC'} > OC$.

Since $\boldsymbol{OB'} = \sqrt{\mathbf{2}}$ and $\boldsymbol{OC} = \sqrt{\mathbf{3}} \times \sqrt{\mathbf{3}} = \mathbf{3}$ have been coming from the straightedge and compass construction, we cannot accept the Pythagorean approximate arithmetic measurement of $\boldsymbol{OB} = \sqrt{\mathbf{3}}$ and $\boldsymbol{OC'} = \sqrt{\mathbf{3}} \times \sqrt{\mathbf{3}} = \mathbf{3}$ as a geometric entity.

5.3 Verification of the length of the hypotenuse and its square for a unit square after an arithmetical approximation of $\sqrt{\mathbf{2}}$ by rounding off at the down

In our previous drawings, the Pythagoras hypotenuse's length has caught on to be longer than the length of the classical construction. Sometimes, irrational and incomplete lengths have to be truncated at the desired decimal place arithmetically for demonstration in the explanation of measuring. Thus, one can automatically estimate the length of the Pythagorean hypotenuse to be smaller than the length of its original algebraic value during his arithmetical approximation location. However, we consider the figure below.

Figure 12
Verification of Pythagoras Theorem for $1^2 + 1^2 = (\sqrt{\mathbf{2}})^{\mathbf{2}}$ *when* $\sqrt{\mathbf{2}}$ *is truncated*



In Figure 12, The Pythagoras Theorem produces $OB = \sqrt{2}$. We take an arithmetic approximation of $\sqrt{2}$ at $(OB - \varepsilon)$, with a similar conception described in 5.2.2. Draw a circle C_1 centering O and radius $(OB - \varepsilon)$. Circle C_1 intersects the line OE at point J ∴ Pythagoras' approximation says that $OJ = \sqrt{2}$. But, while a circle C_2 is drawn, under the classical construction of $\sqrt{2}$, it does not pass through the point J. It is seen that circle C_2 passes through a point J' on the line OE very near point J, where $OJ' > OJ$, and looks like microscopically $OJ < OJ'$. Again, if we construct $\sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}}$ based on the Pythagoras approximate hypotenuse $\boldsymbol{OJ} = \sqrt{\mathbf{2}}$ using straightedge-compass, it produces a length $\boldsymbol{OE}'$, which is smaller than the geometric original length $\boldsymbol{OE} = \mathbf{2}$, i.e., $\boldsymbol{OE}' < OE$.

So, also, in this type of approximation case, we cannot accept the Pythagorean approximate arithmetic measurement of $\boldsymbol{OB} = \sqrt{\mathbf{2}}$ and $\boldsymbol{OE}' = \sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}} = \mathbf{2}$ as a geometric entity.

Note: For Figures 9, 11, and 12, one can argue that since circles $\boldsymbol{C_1}$ have drawn with radii greater and less than $\boldsymbol{OB}$, the amounts of $\sqrt{\mathbf{2}}$ and $\sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}}$ are greater and less. In that case, we would like to say that $\boldsymbol{OB}$ should be taken as such because the Pythagoras Theorem based on an inexact length $\boldsymbol{OB}$ states that the area of $\boldsymbol{OBEG}$ is $\boldsymbol{C}^{\mathbf{2}} = \mathbf{2}$, so the approximation is compulsory here. It is also noted in Figures 9 and 12 that the Pythagorean area of the square $\boldsymbol{OBEG}$ based on the hypotenuse $\boldsymbol{OB}$ of a unit square is twice the unit square or halves of the square $\boldsymbol{HADF}$ was the outcome of the empirical procedure only.

5.4 Findings from the classical geometric construction, incalculable abstract algebraic and approximate arithmetic measurement of the Pythagoras hypotenuse

From section 5, we want to conclude that, classical geometric construction with only straightedge and compass can construct $\sqrt{2}, \sqrt{3}$, …., etc. as truth geometric objects, and their squares, $\sqrt{2} \times \sqrt{2}, \sqrt{3} \times \sqrt{3}$, …., etc. geometrically can be produced accurately, as 2, 3, …., etc. because these are the geometry of length, not an algebraic assumption or an arithmetic approximation. Here, drawing lengths are visible and accurate in the classical construction, e.g., we can divide a unit length into three equal parts followed by classical construction, which is always expressed by line segment, where algebra does not finish this length uttered by an arithmetic number except for approximation or symbolization. If we proceed to achieve $\sqrt{2}, \sqrt{3}$, and their squares $\sqrt{2} \times \sqrt{2}, \sqrt{3} \times \sqrt{3}$ following the Pythagoras algebraic lengths $\sqrt{2}$ and $\sqrt{3}$ for hypotenuse measurement as their approximate arithmetic values, then it is seen that the result does not satisfy the length of the classical constructions. Again, to measure the respective lengths accurately with specific digits from truth geometrical constructions, we cannot avoid the incommensurable location of endpoints of the hypotenuse. So, the Pythagorean formula cannot measure the length of the hypotenuse of a unit square in a rigorous sense. Moreover, if legs are equal, legs (at least one) are irrational, i.e., when $\sqrt{a^2+b^2}$ is not a perfect square number, then the Pythagoras Theorem is not applicable rigorously, but it can approximately.

Although in case of the length of the hypotenuse of a right triangle is an irrational number, the Pythagorean Theorem converts it to a rational number in practical instances. Whether the number is a complete square or not, its square root can be constructed classically as a geometric length that can meet a point on the number line. But, the interesting matter is that all these numbers could not be retrieved conversely from geometric construction because $\sqrt{2}$ is an incomplete number. So, the square root of two exists only in geometric length, i.e., it is possible to get a length in the geometric construction for the symbolic number $\sqrt{2}$ of arithmetic. The actual length $\sqrt{2}$ cannot be found in the geometric construction from the rationalization of an irrational number. Also, other constructions based on it produce errors.

Comment

In section 5, we have observed whether the practical value of the hypotenuse obtained using the Pythagorean formula and the value of an arithmetical operation can return corresponding values inversely to the classical geometric constructions. As a result, we have not been successful in some cases. The Pythagorean Theorem gives the complete and accurate result if a^2+b^2 is a perfect square number. Still, if not a perfect square number, it pushes us to solve another problem related to determining the square root of non-perfect square numbers that is definitely beyond our ability. We think the direction of a theorem shouldn't be such. Pythagoras' statement is hindered here by the compulsion of arithmetical approximation of the length of its hypotenuse to be a theorem. Therefore, we think the world's mathematical authority can revise this area-related statement in the history of mathematics in terms of approximation.

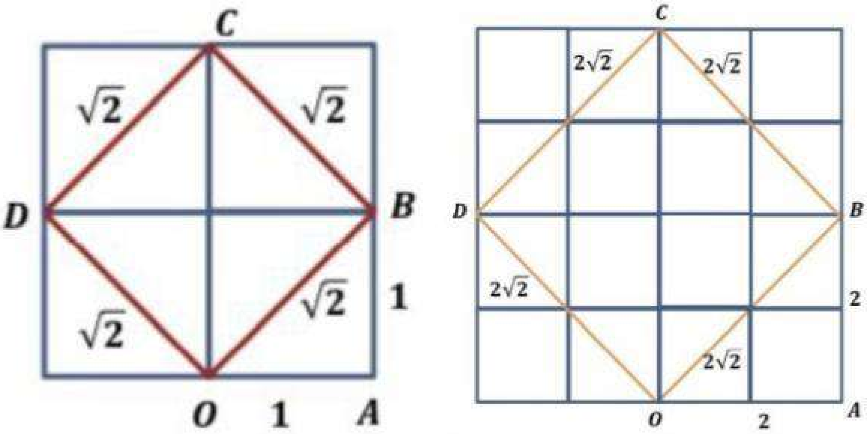
6. Discussion

Although in our present description of plane geometry, the classical construction of the square root of real numbers and the length of the Pythagoras hypotenuse can relate to the same item of the properties of similar triangles, classical construction directly comes from this logic. In contrast, the same reasoning is imposed later for gaining the proof of the Pythagoras Theorem, which is related to the approximate arithmetic outcome of an area.

6.1 Area-based Pythagoras Theorem's outcome: We consider the series of figures below:

Figure 13, 14

Area of Pythagoras' thinking



In the above Figures 13,14, legs are equal and rational on which the sum of areas of squares are not perfect square numbers. It is a simple relation for area measurements from the figures because OB divides the respective squares into two equal parts. So, the sum of the areas on both legs on the isosceles triangles OAB are the same area $OBCD$ on the diagonal OB. Therefore, Pythagoras concluded the algebraic relationship for the areas of squares is $OA^2 + AB^2 = OB^2$. Next, we mathematics practitioners calculate the length of the hypotenuse of a right triangle by taking the square root of OB^2 from the algebraic point of view (Giaffredo, 2015). We want to say that the endless and incomplete algebraic procedure of determining the square root of OB^2 does not match the geometric original length when OB^2 is not a perfect square number. One important concept we must consider is that the length measurement in this situation is not a matter of area calculation. It is a problem of square root extraction for a non-perfect square number. We think the classical geometric (length of an object), theoretical algebraic (abstract), and practical arithmetic (square root approximation) measurements of length $\sqrt{2}$ are not the same, which we will discuss in the next section.

6.2 Pythagoras' approximate arithmetic $\sqrt{2}$ versus geometric length $\sqrt{2}$ from classical construction in a practical sense

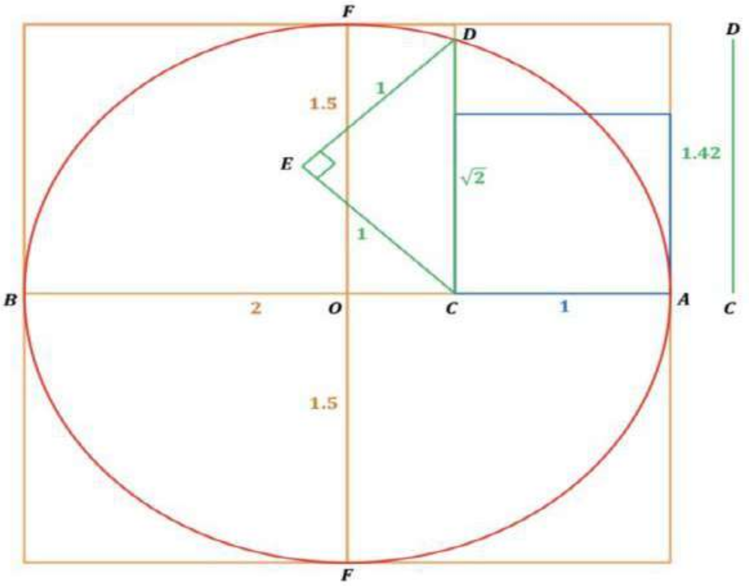
By definition, 2 is not a perfect square number. So, its square root is an approximate and inexact number, and we should use its result by $\sqrt{2}$ as a symbolic number for theoretical purposes and by 1.42 or 1.4142, or at our convenient for practical purposes. But, unfortunately, unconsciously and traditionally, we use $\sqrt{2}$ as the calculating numeric and exact value, which is a common and great mistake. We cannot reach the practical phenomenon until and unless we take the achieving value of it approximately. For example, without taking the approximate value of π, we can never understand the numeric length (circumference as a closed path) and area of a unit circle only symbolized by 2π units and π sq. units. Usually, in an abstract sense, we say that $\sqrt{2}$ is a value whose square is 2. Indeed, what is the fact if we proceed practically? Suppose $\sqrt{2}$ is rounded off at $1.4142136 \therefore (1.4142136)^2 = 2.00000010642496 > 2$. Again let, $\sqrt{2}$ be truncated at $1.4142135 \therefore (1.4142135)^2 = 1.99999982358225 < 2$. Moreover, if we truncate $\sqrt{2}$ at $1.4142135623730 \therefore (1.4142135623730)^2 = 1.999999999999731 < 2$.

Since the Pythagoras Theorem describes an algebraic relationship (Mathematics Staff of The College, 1956), in a practical sense, there is no number in our experience whose square is 2, exactly. Classical geometric construction produces a geometric length whose multiplication with itself is 2, which we can also locate in the Real number line from geometric length, not from algebraic approximate value. But, we cannot extract this square root from the number line without approximation. Therefore, Pythagorean constant $\sqrt{2}$ does not match a specific rigorous arithmetic number except for approximation (Brian Clegg, 2011). Classical construction can make geometric length $\sqrt{x}, x > 0$, which also meets the Real number line at a point, but all of these cannot be read out arithmetically (root extraction) in an accurate sense. An article (Kinney, 2019) says, '*Doesn't the diagonal of a square indeed have a length? Isn't the Pythagorean Theorem true*?' Yes, a diagonal have length, but from the number line, every length we cannot convert accurately into a specific number. Secondly, Pythagoras Theorem is true for $a^2 + b^2$ being a perfect square number and rigorously no for $a^2 + b^2$ being not a perfect square number.

6.2.1 Comparison of our drawings with Microsoft software drawings

In section 5, we drew some geometrical figures by hand with a straightedge and compass completely, abiding by the principle of classical construction. After the drawing, we found the microscopic mismatches between the length measurements under classical constructions and arithmetic approximation. The Pythagoras hypotenuse length is measured simply by joining two endpoints of the legs for each equal (rational) and unequal (rational and irrational) leg, then automatically taking the practical approximate arithmetic value. Now, we are going to compare the quality of our hand drawing and check the validity of our previously observed geometric and approximate arithmetic $\sqrt{2}$-related length measurements with the measurements in Microsoft software (the drawing has been completed here below in a Microsoft Word document along with taking the necessary tools from the Insert tab) in the following Figure 15.

Figure 15
Construction of $\sqrt{2}$ with Microsoft software



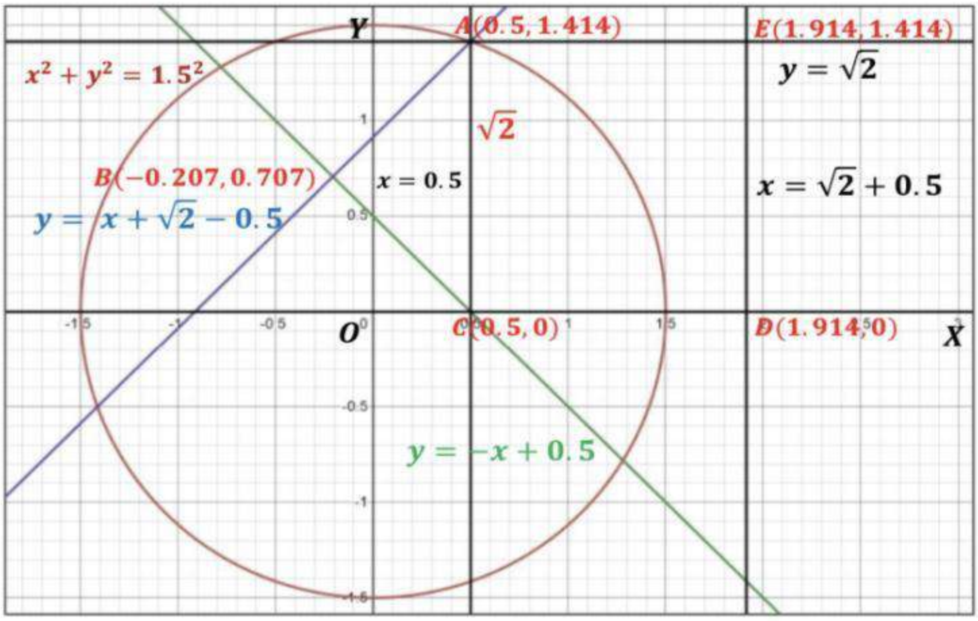
In Figure 15, $OA \perp OF, BC = 2, CA = 1, CD \perp CA, OA = OB = OE = OF = 1.5$. A circle has been drawn here with center O and radius OA, which intersects CD at D. Therefore, in the geometrical construction of numbers here, a number $CD = \sqrt{2}$ is produced where $CD \times CD = 2$.

A horizontal right triangle CED with legs $CE = DE = 1$ is drawn and rotated 45° clockwise. The hypotenuse of the right triangle has shifted along line CD and the Pythagoras hypotenuse coincides with the formerly constructed $CD = \sqrt{2}$ with the length. So, the length of $\sqrt{2}$ from geometrical construction and the length of the hypotenuse of a right triangle seem to be the same. But, by inserting a vertical line segment aiming to scale, we find that the object's (hypotenuse) height measured by the system is $CD = 1.42$ as if $CD^2 = 2.0164 > 2$. The approximate arithmetic measurement of Pythagorean hypotenuse $\sqrt{2}$ is affected here by itself because of its endless behavior. In Figure 9 and Figure 11, we found that the length of the Pythagorean approximate hypotenuse was longer than the construction length, which is similar here. So, our previous measurements are satisfied by this new drawing. Whether we want it or not, the system automatically approximates $\sqrt{2}$. Therefore, Figure 9 and Figure 11 are valid drawings.

6.2.2 Comparison of our drawings with GeoGebra software drawings

The Pythagoras formula and Cartesian coordinate system are the same idea in different contexts. Although the Pythagoras formula has numerous proofs, it has become a blind tool in our math learning and teaching because most (58% out of 43 participants) of us involved in teaching math are not used to setting our lesson plans by including the proof regarding the Pythagoras Theorem (Güner, 2018). GeoGebra software, based on Pythagoras formula and Cartesian coordinate system, is a more effective and popular software to practice and understand algebra, geometry, calculus, etc. Construction of a regular polygon on three sides of a right triangle through GeoGebra used to generalize the Pythagoras Theorem (De Sousa et al., 2023). For a better understanding of the Pythagoras Theorem as the compulsion of an arithmetic approximation for the hypotenuse of a unit square, we can consider the GeoGebra-related figure below.

Figure 16
Construction of $\sqrt{2}$ *with GeoGebra software (From Desmos Calculator)*



In Figure 16, $x^2 + y^2 = 1.5^2$ is a circle with a center of $(0, 0)$ and a radius of 1.5, and $x = 0.5$ is a straight line perpendicular to the x-axis at the point C. For making an isosceles right triangle ABC with sides $AB = BC = 1$, we plot two perpendicular Straight lines, BC and AB, whose equations are $y = -x + 0.5$ and $y = x + \sqrt{2} - 0.5$, respectively. For making a square $ACDE$ on the hypotenuse AC, set three straight lines, $y = 0$, $y = \sqrt{2}$, and $x = \sqrt{2} + 0.5$.

GeoGebra itself creates the coordinate of the intersecting point of the circle and straight line as $A(0.5,\ 1.414)$ and the intersecting point of the straight lines AB and BC as $B(-0.207,\ 0.707)$. Similarly, $C(0.5,\ 0)$, $D(1.914,\ 0)$, and $E(1.914,\ 1.414)$ have been found here. Here, we have no scope to approximate any length or point manually. Compared with our previous classical construction, here we have $CA = \sqrt{2} = 1.414$, whose square is $1.999396 < 2$, which violates the existence of $\sqrt{2}$. Straight lines BA passes through point A, BC passes through point C. Following the Pythagoras and Cartesian formula for achieving the distance between two points on the plane, we get $AB = a = \sqrt{0.999698} \neq 1$, $BC = b = \sqrt{0.999698} \neq 1$, which violates the right angled triangle ABC with unit sides.

Again, if we take the area of the square $ACDE$ based on the hypotenuse $CA = \sqrt{2}$, we get the area $= 14 \times 14 \times (0.1 \times 0.1) + (14 + 14) \times 0.1 \times (1.414 - 1.4)$ sq. units $= 1.9992 \neq 2$ sq. units, which violates the Pythagorean algebraic amount of the area $c^2 = a^2 + b^2 = 2$. The radius of the circle here is $OA \neq 1.5$.

For the above discussion, some questions may arise, which are as follows:

Didn't we start the process in the classical construction to get $\sqrt{2}$? Was not the distance of the straight line whose equation is $y = x + \sqrt{2} - 0.5$ from point $C(0.5, 0) = \frac{0.5 - 0 + \sqrt{2} - 0.5}{\sqrt{2}} = 1$, and of the straight line whose equation is $y =- x + 0.5$ from point $A(0.5, \sqrt{2})$ was not $\frac{0.5 + \sqrt{2} - 0.5}{\sqrt{2}} = 1$? Isn't the square $ACDE$ formed based on side $\sqrt{2}$?

Mathematics readers are sometimes confused to identify the greater or smaller values regarding square root (İşleyen & Mercan, 2013), e.g., which of $3\sqrt{17}$ and $4\sqrt{13}$ is the greater numerical value? To properly understand this, we have to take the approximate values as $3\sqrt{17} = 12.369$ and $4\sqrt{10} = 12.649$, where both of these are smaller than their original values. All this created a chaotic situation. An acceptable answer is that if the decimal places of approximation increase, the error will decrease. However, it is never possible to be 100% error-free in the numerical value of the length.

Only blind believing thinking in the abstract sense can accept the square root of two or the Pythagorean hypotenuse $c = \sqrt{2}$ as an exact physical number. We want to say that such approximation can be for general calculation but should not be attached to a prominent theorem like Pythagoras. Compared with Figure 12, GeoGebra matches the measurement of Pythagorean hypotenuse $\sqrt{2}$, which is smaller than the geometric original length of $\sqrt{2}$, but makes a remarkable error in the case of unit sides with our previous classical construction because GeoGebra has to follow the approximation for demonstrating a value. All numbers and points in the graph of GeoGebra are rational. None of the irrational can be discussed or visualized in this media. So, to discuss or to understand the existence of an irrational number like $\sqrt{2}$ in a rigorous sense through GeoGebra is not suitable, we think. In practical terms, whether we like it or not, the system will force convergence of the arithmetic value so that the Pythagorean hypotenuse $c \approx 1.41$ or 1.42 or something like this for equal unit legs.

Comment

Algebraic abstract number $\sqrt{2}$, arithmetic approximate number $\sqrt{2}$, and geometry (the length of a geometrical object) about $\sqrt{2}$ are different here. Therefore, the length $\sqrt{2}$ of the hypotenuse can only be constructed in geometry by the straightedge-compass principle as the length of a line segment. It cannot be exactly completed by doing the Pythagoras algebraic approximate calculation or Cartesian distance $c = \sqrt{a^2 + b^2}$.

6.3 Three proofs regarding the square root of two

The three indirect algebraic proofs exactly (with some elaboration) have been taken from web portals (Wikidot, n.d.; Mathematics, n.d.; Wikipedia, n.d.-d) and discussed below:

Theorem 1: There is a positive real number $x = \sqrt{2}$ such that $x^2 = 2$

Proof of Theorem: *Let the set S be such that $S = \{s: s \in R$ and $0 \leq s, s^2 < 2\}$. We first note that this set is nonempty since clearly $1 \in S$ that is $0 \leq 1$ and $1^2 = 1 < 2$. Furthermore, we note that this set is also bounded above by 2 since for any $t \in R$, if $t > 2$ then $t^2 > 4$ so $t \notin S$.*

Therefore this set has a supremum in R. Let $x =$ Sup S and $x > 1$. We will now show that our supremum $x = \sqrt{2}$ by showing that $x^2 < 2$ and $x^2 > 2$ are invalid cases.

First consider the case where $x^2 < 2$. We want to find a natural number $n \in N$ such that $(x + \frac{1}{n}) \in S$ which would imply that $x =$ Sup S is not an upper bound.

$$(x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{2x}{n} + \frac{1}{n} = x^2 + \frac{1}{n}(2x + 1)$$

Now we want to choose $n \in N$ such that $x^2 + \frac{1}{n}(2x + 1) < 2$ or rather $\frac{1}{n}(2x + 1) < 2 - x^2$ so that $(x + \frac{1}{n})^2 < 2$. Now we note that $\frac{2 - x^2}{2x + 1} > 0$ since the numerator is positive because $x^2 < 2$ so $0 < 2 - x^2$ and the denominator is positive since $x \geq 0$. Therefore, by The Archimedean Property, there exists some natural number n such that $0 < \frac{1}{n} \leq \frac{2 - x^2}{2x + 1}$. Therefore we have obtained such a number $n \in N$ where $(x + \frac{1}{n}) \in S$ which contradicts the fact that $x =$ Sup S since $x < x + \frac{1}{n}$.

Now consider the case where $x^2 > 2$. *We want to show that it is possible to find a natural number* $m \in N$ *such that* $(x - \frac{1}{m})$ *is also an upper bound of S. Now note that:*
$(x - \frac{1}{m})^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$

So we want to choose $m \in N$ *such that* $x^2 - \frac{2x}{m} > 2$ *or equivalently* $\frac{2x}{m} < x^2 - 2$ *to ensure that* $(x - \frac{1}{m})^2 > 2$. *We know that* $x^2 - 2 > 0$ *and* $2x > 0$ *and so therefore:*

$$\frac{1}{m} < \frac{x^2 - 2}{2x}$$

Once again by the Archimedean Property we know that such a natural number m *exists. Now suppose that* $s \in S$. *Then* $s^2 < 2 < (x - \frac{1}{m})^2$ *which implies that* $(x - \frac{1}{m})$ *is an upper bound for S. But then we have a contradiction since* $x = Sup\ S$ *is not the supremum of S since* $x - \frac{1}{m} < x$. *Therefore our only possibility is that* $x^2 = 2$ *and so* $x = \sqrt{2} = Sup\ S$. *Since the supremum of a set is a real number, we conclude* $\sqrt{2} \in R$.

Our response regarding the above proof

For this proof, first, we let $x^2 = y$ and $z = 2$. The law of the trichotomy states that if y and z are any two real numbers, then one and only one of the following three is hold: $y < z, y = z, y > z$. In the proof, the logic taken from the Law of trichotomy is either $x^2 < 2$ or $x^2 = 2$ or $x^2 > 2$. Although the statement contains two parts, where one part is the existence of a real number $x = \sqrt{2}$ and the other part is a square of it, i.e., $x^2 = 2$, the proof has already admitted the existence of a Real number $x = \sqrt{2}$ as a completed number, then completed the evidence of the second part $x^2 = 2$. However, we will enjoy the proof of the existence of the irrational Real number $x = \sqrt{2}$ in the following theorem 2. On the other hand, in the trichotomy (Wikipedia, n.d.-b), it is declared that "*In classical logic, this axiom of trichotomy holds for ordinary comparison between real numbers and therefore also for comparisons between integers and between rational numbers. The law does not hold in general in intuitionistic logic.*" So, it does not make sense why we consider critical comparison on the symbolic number $\sqrt{2}$, which is an incomplete arithmetic number under the incommensurable operation of the square root of a non-perfect square number. The metaphor seems "We are going to issue a character certificate for $\sqrt{2}$, but we have already admitted $\sqrt{2}$ as a model character." It assumes that the procedure mentioned in the above proof can be the same for any Real number, e.g., there exist real numbers x such that $x^2 = 3, 4, 9, \ldots$, etc. The result of the above proof must occur at any open interval. If we consider an open interval (a, b), then for some $n \in N$, we will get $(x + \frac{1}{n}) \in (a, b) \forall x < b$ and $(x - \frac{1}{n}) \in (a, b) \forall x > a$.

What will happen if we demonstrate the proof of theorem 1 in the following alternative ways regarding the existence of the Real number x such that $x^2 = 2$:

Alternative 1:

Respective set is $S = \{s : s \in R \text{ and } 0 \le s, s^2 < 2\}$
Let $x = \text{Sup } S$ and $x > 1$.
Consider the case where $x^2 < 2$.
Now $(x - \frac{1}{n})^2 < (x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} < x^2 + \frac{2x}{n} + \frac{1}{n} = x^2 + \frac{2x+1}{n}$ (1)

Here, $\frac{2 - x^2}{2x + 1} > 0$ since the numerator is positive because $x^2 < 2$, and the denominator is positive since $x > 1$.

Therefore, by The Archimedean Property, there exists some natural number $n \in N$ such that $0 < \frac{1}{n} < \frac{2 - x^2}{2x + 1}$. $\therefore \frac{2x + 1}{n} < 2 - x^2$ or $x^2 + \frac{2x + 1}{n} < 2$ (2)

From (1) and (2), we get $(x - \frac{1}{n})^2 < 2$

Therefore, we have obtained such a number $n \in N$ where $(x - \frac{1}{n}) \in S$, which is valid. So, $x^2 < 2$ can be the square of the Sup $S = x$.

Alternative 2:
Respective set is $S = \{s: s \in R \text{ and } 0 \leq s, s^2 < 2\}$
Let $x = \text{Sup } S$ and $x > 1$.
Consider the case where $x^2 > 2$.
Now $(x + \frac{1}{m})^2 > (x - \frac{1}{m})^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$ (3)

Here, $\frac{x^2 - 2}{2x} > 0$ since the numerator is positive because $x^2 > 2$, and the denominator is positive since $x > 1$.

Therefore, by The Archimedean Property, there exists some natural number $m \in N$ such that $0 < \frac{1}{m} < \frac{x^2 - 2}{2x}$. $\therefore \frac{2x}{m} < x^2 - 2$ or $x^2 - \frac{2x}{m} > 2$ (4)
From (3) and (4), we get $(x + \frac{1}{m})^2 > 2$.

This implies that for such a number $m \in N$, $(x + \frac{1}{m})$ is an upper bound for S, which is valid. So, $x^2 > 2$ can be the square of the Sup $S = x$.

Alternative 3:
The capacity of our calculator says $\sqrt{2} = 1.414213562373095$, $(1.414213562373095)^2 = 2$. Since here $(1.414213562373095)^2 = 2$, so, $1.414213562373095 > s, \forall s \in S$.
Now, for $n = 9000000000000009 \in N$, we get $(x + \frac{1}{n}) = 1.414213562373095 \notin$ S.
So, $x = 1.414213562373095$ is the least upper bound of S.

Again, for $n = 9000000000000009 \in N$, we get $(x - \frac{1}{n}) = 1.414213562373095$, which implies that the least upper bound remains unchanged.
Therefore, the Sup $S = x = 1.414213562373095$, proves the existence of the Real number $x = \sqrt{2}$ such that $x^2 = 2$.

From the above discussion in this section, can we say that x is the supremum of S such that $x^2 < 2$ or $x^2 > 2$, whereas $x = \sqrt{2}$ has been proved the supremum of S such that $x^2 = 2$ in Theorem 1? No, we cannot say that because these proofs do not give strictly accurate results due to the imperfection of irrational numbers. Alternative 1 and 2 are biased partial truths because the contradictions $(x + \frac{1}{n}) \in S$ and $(x - \frac{1}{n})$ is an upper bound of S, which we mentioned in Theorem 1. Alternative 3 says a rational number $x = 1.414213562373095$ is the supremum of S such that $x^2 = 2$, but we know that there exists no rational number $r = \frac{a}{b}$ such that $r^2 = 2$, which will prove in Theorem 3. Rather than biased processes, arithmetic flexible approximation we can introduce here. Therefore, there exists a real number $x \approx 1.414213562373095$ such that $x^2 \approx 2$.

Theorem 2: $\sqrt{2}$ is an irrational number.

Proof: *Suppose that $\sqrt{2}$ is a rational number and $\sqrt{2} = \frac{p}{q}$, where $p, q \in Z, q \neq 0$, and $(p, q) = 1$*
So, $p^2 = 2q^2$
Here, p is an even number because its square is even.
Let, $p = 2r$.
So, $q^2 = 2r^2$
Similarly, q is also an even number.
Contradiction has been arisen that $(p, q) = 1$.
Therefore, $\sqrt{2}$ is an irrational number.

Our response regarding the above proof

In the above proof, we admitted that $(\sqrt{2})^2 = 2$ as if we know $\sqrt{2}$ in complete form, which is the main issue. Theorem 1 has been proved based on Theorem 2 and vice versa. To reflect the existence of factors in our mind without ambiguity and rational acceptance like the Theorem, we will first have to prove, yes, there is an algebraic complete number $\sqrt{2}$, then its square $(\sqrt{2})^2 = 2$, like $\sqrt{4} = 2$ to $(2)^2 = 4$. But here, unfortunately, a kind of classical ambiguity has to be observed in every effort concerning the proof of algebraic irrationality.

Theorem 3: There exists no rational number $r = \frac{a}{b}$ $(a, b \in Z$ ***and*** $b \neq 0)$ ***such that*** $r^2 = 2$.

Proof of Theorem: *We will do this proof by contradiction. Suppose that* r *is a rational number in lowest terms, that is the integers* a *and* b *have* 1 *as their greatest common divisors commonly abbreviated as gcd* $(a, b) = 1$, *and* r *is such that* $r^2 = 2$. *Therefore:*

$$r^2 = 2 \qquad (1)$$

$$(\frac{a}{b})^2 = 2$$

$$a^2 = 2b^2$$

Therefore it follows that a^2 *is even which implies that* a *is even. Since* a *is even it can be written as* $a = 2m$ *for some* $m \in Z$. *Making this substitution we get that:*

$$a^2 = 2b^2 \qquad (2)$$

$$(2m)^2 = 2b^2$$

$$4m^2 = 2b^2$$

$$2m^2 = b^2$$

And we see that b^2 *is also even which implies* b *is even so* b *can be written as* $b = 2n$ *for some* $n \in Z$. *Since* $a = 2m$ *and* $b = 2n$ *for some* $m, n \in Z$, *it follows that gcd* $(a, b) =$ *gcd* $(2m, 2n) \geq 2$ *which contradicts our original assumption that* r *was rational.*

Our response regarding the above proof:

We can consider it a comparatively reasonable proof among the above three theorems, although the source of the conception is the same. From the beginning of the concept of irrational numbers, including the earliest number $\sqrt{2}$, we have been saying what they are not rather than what they are to understand how they became numbers (Cepelewicz, 2024). German mathematician Julius Wilhelm Richard Dedekind (1831 – 1916) gave his famous Dedekind cut for the definition of Real numbers to maintain the continuity of Real numbers. In one place in his writing (Wikipedia, n.d.-a), Dedekind says, *'Note that the equality* $b^2 = 2$ *cannot hold since* $\sqrt{2}$ *is not rational.'* Dedekind Cut gives us a comparatively reasonable explanation regarding irrational numbers, but it still hasn't and never will quench the thirst of mathematicians. It is easier to admit irrational numbers as approximations or to take them as axioms but less likely to take them strictly as absolute algebraic numbers. Based on the sequence of calculation, we expect the calculation of $(\sqrt{2})^2$ to go in such that it can become arbitrarily close to 2, but nature's sign is that it never goes to completeness. Therefore, rigorously, we cannot conclude that $(\sqrt{2})^2 = 2$ from an algebraic point of view.

6.4 Speaking to rigorous truth

The author of this paper is searching for the answer to the following question: We can approximate any arithmetical calculation to overcome its unboundedness with imperfections, but when a mathematical statement undergoes by force into such approximation processes, can the statement be accepted as a mathematical theorem in the principle of mathematical rules and regulations or axiom? A quotation from the Philosophy of mathematics is given below (Wikipedia, n.d.-c):

> "*Mathematical reasoning requires rigor. This means that the definitions must be absolutely unambiguous and the proofs must be reducible to a succession of applications of syllogisms or inference rules, without any use of empirical evidence and intuition.*"

We have tried to show in the discussion of this paper that the practical value of the Pythagorean hypotenuse (in the case of irrationality) has to approximate every time. Conversely, this practical value can no longer adapt to the exact geometric length of the hypotenuse. One thing is noteworthy that the theorem that Pythagoras gave in favor of his own belief only in the existence of rational numbers was then initially arrested by the existence of other types of numbers by Hippasus, so why did the statement remain valid as a theorem? *Mathematical theories must be based on axioms (basic assumptions that are considered as true)* (Wikipedia, n.d.-c). We think the matter could have settled based on whether or not $a^2 + b^2$ is a perfect square. *In deductivism, the Pythagorean theorem is not an absolute truth, but a relative one, if it follows deductively from the appropriate axioms* (Wikipedia, n.d.-c). Later, many great mathematicians argued for the existence of irrational numbers, showing us that the Pythagorean Theorem also applies to irrational cases. If it works, why are we rationalizing the irrational? It is futile to argue that by representing incomplete irrational numbers as completed by symbols, all will accept unexpressed truths as revealed truths. We have no experience or example to expect that square roots of non-perfect square numbers will one day be completed numbers. Is it a similar case that we add fetuses to the current population count by knowing fetuses developing in many mothers' wombs? Have there been symptoms in the past three thousand years that we understand where $\sqrt{2}$ is waiting to end? We realize that it will never finish. So, we should give credibility to mathematics by treating these fields as approximations rather than absolute truths.

The question may arise then, is the geometric length $\sqrt{2}$ false?" No, the geometric size $\sqrt{2}$ is a visible fact whose algebraic real value is beyond our ability to determine. We can locate only a point for $\sqrt{2}$ on the Real number line, but the arithmetical identity of this point has not yet been told accurately or extracted in particular digits. We cannot fully understand its true character, or there is no symptom that we can understand it completely. It remains unbound and incommensurable. The definition of the square root of a number states that the square root of x is such y that $y^2 = x$. Now, the simple thing is that this y must be determined correctly first, and then see if $y^2 = x$. If we do not do so and accept it by resorting to symbolic tricks like superstition, it will be to hide the truth. Almost truth and complete truth are not the same thing - if we avoid this fact, we call the axiom of geometry into question. By avoiding the truth, we are trying to make everyone interested in gaining credibility by accepting $\sqrt{2}$ as the truth value of the square root of 2. We have developed many branches of mathematics based on these attempts to adapt, which have enhanced the beauty of mathematics but also given rise to debates about deviations from truth. In other branches of science, working with the results obtained from this approximation process may not cause so much harm, or the amount of error in the corresponding calculation is so low that it can be considered negligible for the purpose. However, according to the scriptures, the truth should be accepted as the entire truth. Otherwise, small mistakes will lead to big mistakes.

The proofs by contradiction are not transparent to us in various statements regarding the square root of 2. For example, to prove that $\sqrt{2}$ is an irrational number while making $a^2 = 2b^2$, by assuming $(\sqrt{2})^2 = 2$ whereas we do not know accurately the value of $\sqrt{2}$. The argument derived from the definition of square root here is not clear. Can we not counter-argument that the square root of all non-perfect square real numbers is undecidable because there are no real ones in our calculation ability whose square is 2 except in the notation $\sqrt{2}$, as seen in the case of the square root of negative numbers? It does not make sense to us how can we know $(\sqrt{2})^2 = 2$ while determining the arithmetic identity of $\sqrt{2}$. Again, proving $(\sqrt{2})^2 = 2$ with the existence of a Real number $\sqrt{2}$ in Theorem 1 in subsection 6.3 takes three alternatives for the square of the supremum of a set S, first checking the possibility of having supremum of S in two invalid sets $x^2 < 2$ and $x^2 > 2$. The two corresponding sets are invalid because, since such a result must occur for open sets, it does not seem necessary to prove these. The third alternative, $x^2 = 2$, mentioned in the proof, is pre-planned, biased, and unreasonable. Here, it assumes that the square of the supremum of the set S can be 2, i.e., the square of a Real number can be 2, although it is our original issue. We also see in the proof that the supremum of S is declared $\sqrt{2}$, which means that the existence of the square root 2 is accepted only based on the incomplete algebraic operation and the symbolic expression $\sqrt{2}$. These theorems perhaps have been prepared based on the logic of blind imitation, and the two visible things covered in the proof are:

1) The definition of the square root is applied so that it is not necessary to know what the square root of 2 is. Whatever the square root of 2 is, its square will be 2 because we are talking about the

square root of 2. The point is that it doesn't matter what the square root of any number is. Per contra, the square of the square root of positive x will be x.

2) The square root of 2 is $\sqrt{2}$, i.e., the square root of positive x will be $\sqrt{x}$. It seems that no arithmetic calculations are necessary here. Only the symbol $\sqrt{x}$ is the vital tool.

Fundamentally, until we say that $\sqrt{2}$ is a possible known arithmetic number like $\sqrt{2} \approx 1.41421356237309 \therefore (1.41421356237309)^2 = 1.999999999999986 \approx 2$, the doubts and thirsts of math readers will not be satisfied. We think that the mathematical declaration of the Pythagorean Theorem can be revised. *For example, at one time, the Greeks held the opinion that 1 (one) was not a number, but rather a unit of arbitrary length* (Wikipedia, n.d.-c). But now this opinion has been changed. Hence, the beauty of declaration can only be enjoyable based on approximation (for relevant cases), not based on the absolute truth. Absolute truth only can be found in such cases in the classical geometric constructions along with its approximate numerical conversion can be enjoyed in the arithmetic approximations for irrational numbers. All other unexplained symbol-dependent abstract truths are contrary to the Philosophy of mathematics. *David Hilbert held the opinion that there was no other meaningful mathematics whatsoever, regardless of interpretation* (Wikipedia, n.d.-c).

6.5 Last thought

Concepts such as the perimeter of a unit circle and the diagonal (Pythagorean hypotenuse) of a unit square can be constructed classically as specific and visible geometric lengths. These lengths are expressed only by symbols 2π or ◌ (Roy, 2021) and $\sqrt{2}$ but not as particular physical numbers. In solution to real-life-oriented problems, we have to work with approximate standards. If we try to regain the original geometric size from the approximate value of $2\pi = 6.28$ of the circumference of a unit circle, it will not be possible because the value is not 100 percent true. Similarly, an approximation of the Pythagorean unit square's hypotenuse to 1.4142 will not return the original geometric length of that hypotenuse. Although the geometric lengths of such irrational numbers occupy places as points on the Real number line, none can express these in complete numerical values without symbolization or approximation. We can only observe the graceful existence of their geometric lengths. It is not desirable to violate mathematical discipline by making emotional decisions, overwhelmed by the beauty of unbounded numbers. We believe that arguments should be delivered honestly, which can further develop the beauty of mathematics. Since Pythagoras' formula is partially true, it asks for thought to what extent it makes sense to consider it a theorem. We do not treat Archimedes' perimeter and area formulae as theorems for circles. We have tried to demonstrate in our paper that we are facing the inevitable problem of declaring the Pythagorean formula scientifically gives a hundred percent true result. If the learned reader wants to examine it differently, he or she must first bring $\sqrt{2}$ out of its shell, and then he can find what unknown mystery happened in our beloved famous formula. It seems to us failure to analyze the matter from an unbiased point of view will cause pain to many in the mathematical world.

7. Drawbacks of the study and Future research

The Pythagoras Theorem is the most important mathematical tool established on area-based empirical algebraic relation that relies entirely on approximations in the case of irrationality - this might be the gist of this article. Aiming to show which deviation occurs and what happens to a statement without having the specific characteristics of being a mathematical theorem at hand, the same point repeated in our article, which is a limitation of our paper. Also, geometrical mapping has to take the asymmetric position of a point in microscopic distance estimation, which is also the limitation of our study. We hope that our paper will help the junior readers to understand the importance of analyzing the properties of Pythagoras' statement to be a theorem, i.e., yielding a hundred percent true results, and will enable the learned mathematicians to unravel the mysterious behavior of the theorem.

8. Conclusion

We mathematics readers have become accustomed to thinking about rationalizations and symbolizing irrational numbers like the square root of two. The matter stood at such a level that we are saying the length of a geometric object is $\sqrt{121}$ units instead of 11 units. It is not a real-life-based solution process in this symbol system, and understanding the process of real-life-based solutions. We have no other way except for the arithmetical approximation to get out of such symbol publishing, e.g., if we declare the distance between two places is $\sqrt{520}$ kilometers, we cannot guess how far it is indeed. But, if it is said nearly 22.83 kilometers, then the general people can easily understand the distance. In the case of rigorous mathematical study, this estimate will not provide the exact value, but it will assist in understanding the tendency of the quantitative phenomenon. On the other hand, we cannot solve any practical problem with an invisible symbolic value like $\sqrt{2}$. Therefore, we want to believe in its usefulness in working by rationalization through the approximation of irrational numbers to solve practical life-based problems related to exact visible geometric lengths.

We can construct $\sqrt{2}$ as a length or line segment followed by some particular steps with a straightedge and compass, where $C \times C = 2$. Pythagoras Formula was not established based on a mathematical principle (Kalanov, 2013). It was a derived result from an experience regarding the area and algebraic relationship. Numerous proofs based on algebra, geometry, vectors, calculus, and logic have been introduced here in mathematics for Pythagoras Theorem. Even if these relative logical and rational steps greatly enrich the world of mathematics, or even if the Pythagoras Theorem is considered correct in approximation, it does not mean that everyone will think that the Pythagoras Theorem is strictly unquestionable as a theorem. For any calculation regarding practical uses, we should elaborate on $\sqrt{x}$ at our convenience, whether x is a perfect square number or not. We tried to express our logic with the best of our understanding that $\sqrt{2}$ representing line segment can only be constructed geometrically in classical construction, which cannot be uttered arithmetically in complete form as well as from the computed value of the algebraic form of $x^2 = 2$, $\sqrt{2}$ cannot be produced geometrically in classical construction. The Pythagoras algebraic hypotenuse $c = \sqrt{a^2 + b^2}$ in the isosceles right triangle cannot 100 percent ensure the existence of $\sqrt{2}$ because, in this case, $\sqrt{a^2 + b^2}$ is an inexact number. Since the length of the hypotenuse of a unit square is an approximate number, the Pythagoras Formula here describes an inexact number c as symbolized by $\sqrt{2}$ or approximated by 1.42, 1.41, 1.4142, and something like these when $a^2 + b^2$ is a non-perfect square number which contradicts the acceptance of the Pythagorean statement as a theorem. However, we believe that such an issue cannot be resolved smoothly. Only it can be mitigated by approximation with imperfections. So, arithmetic approximation is the only way to mitigate it, and imperfection sometimes enhances the beauty of mathematics (Roy, 2023).

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Conflict of Interest

The author declare no conflict of interest.

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