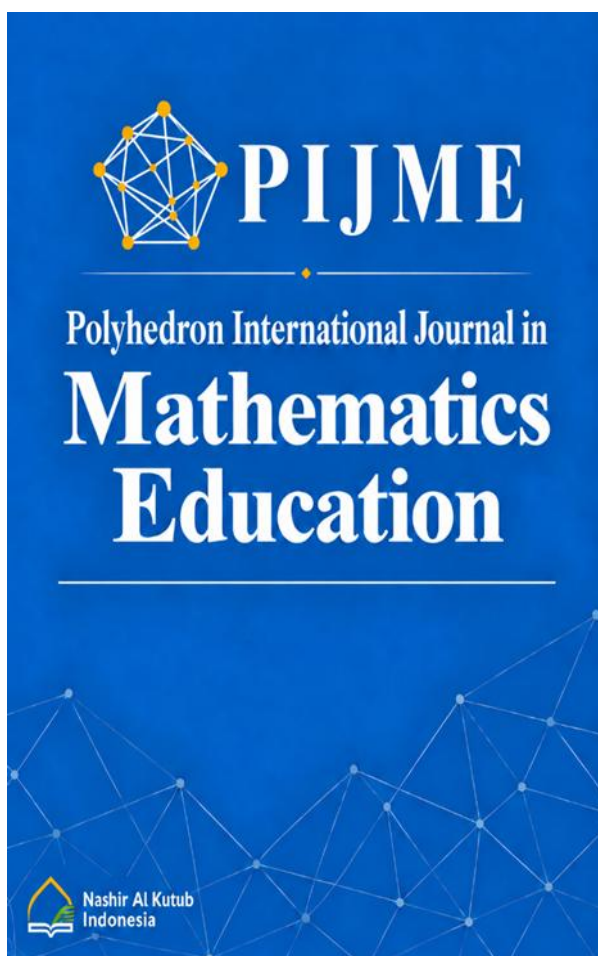




Ethnomathematics of sacred movement: Modelling the spatial geometry of Tawaf and Sa'i through the extended theory of mathematical connections

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Ethnomathematics of sacred movement: Modelling the spatial geometry of Tawaf and Sa'i through the extended theory of mathematical connections

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Abstract

Ethnomathematics research has largely examined static artifacts such as buildings, ornaments, and textiles, while patterned human movement has received little attention. This study analyzed the spatial-geometric and kinematic structure of tawaf and sa'i, the two movement-based pillars of umrah, using the Extended Theory of Mathematical Connections (ETMC) within an emic-etic-dialogical (ethnomodelling) framework. Official manasik guidebooks, scholarly texts, and publicly available visual records were analyzed through document and visual content analysis in three phases: emic description of the prescribed rules, etic geometric and kinematic modelling, and dialogical coding of the connections between them. Tawaf was modelled as a positively oriented closed path with winding number seven around a non-circular convex obstacle formed by the Ka'bah and H̥ijr Ismā'īl; by Steiner's formula, the length of each circuit grows linearly with distance from the Ka'bah, $L(r) = PK + 2\pi r$, regardless of the building's exact shape. Sa'i was modelled as a reversible open path in which the ritual rule of ending at Marwah follows from the parity of seven one-way passages. The shared emic counting unit, shawṭ, denotes a full period in tawaf but a half period in sa'i. Raml and harwalah add piecewise-uniform motion that supports graphs of motion, distance-displacement reasoning, and an optimization model. All three ETMC categories and all seven intra-mathematical connection types were identified. The study extends ethnomathematics to collective ritual movement, offers a replicable procedure for grounding models in emic rules, and proposes a learning trajectory for geometry education in Islamic schools.

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1. Introduction

Mathematics is not only learned in classrooms; it is also enacted in the ways communities build, count, measure, and move. This idea lies at the heart of ethnomathematics, which D'Ambrosio (1985, 2006) defined as the mathematics practiced by identifiable cultural groups and which Bishop (1988a) grounded in six activities found in nearly all cultures: counting, locating, measuring, designing, playing, and explaining. From this perspective, mathematics is a practice that grows out of the needs and the cultural context of a community rather than a single, purely formal body of knowledge (Rosa & Orey, 2011). What ethnomathematics can reveal, however, depends heavily on which cultural practices researchers choose to study.

To date, that choice has favored static objects. In Indonesia, ethnomathematics research has grown rapidly through studies of the Sumur Gumuling structure in Tamansari (Alvian et al., 2021), Yogyakarta batik patterns (Prahmana & D'Ambrosio, 2020), cultural activities in South Sumatra (Araiku et al., 2020), and the traditional architecture of Kampung Kuta (Wijayanti et al., 2026). In the Islamic domain, studies have documented symmetry, transformations, and proportion in mosque ornamentation and geometric patterns (Bonner, 2017; Dabbour, 2012). Such objects are rich in shape, symmetry, and proportion, but they hold what Gerdes (1986) called "frozen" mathematics: the geometry is fixed in the finished product, while the time, direction, and repetition of the activity that produced it are no longer visible. Practices that consist of patterned movement keep these dimensions in view, and few movement practices are as precisely codified as the rites of the umrah pilgrimage.

The umrah contains two such rites. In tawaf, pilgrims circumambulate the Ka'bah seven times; in sa'i, they walk seven times between the hills of Şafā and Marwah. Both rites are governed by explicit rules about where to begin and end, which direction to move, what to count, and where to change pace, and millions of pilgrims from different cultural backgrounds follow these rules every year. Precisely because the rules are codified and shared across cultures, the rites are well suited to systematic mathematical analysis. Yet existing scholarship has examined tawaf and sa'i mainly from the perspectives of Islamic jurisprudence (fiqh), pilgrimage management, pilgrim health, and crowd engineering, and within the literature we reviewed, we did not find a study that analyzes their mathematical structure from the perspective of ethnomathematics or mathematics education.

This gap matters for both theoretical and pedagogical reasons. Theoretically, ethnomathematics needs to diversify its cultural contexts to remain a living research program (Rosa & Orey, 2016), and movement-based rites can show whether ideas such as orientation, cyclic counting, and rate of change are embedded in cultural practice in the same way that symmetry and proportion are embedded in artifacts. Pedagogically, closed and open paths, orientation, parity, and motion are taught from elementary

school to teacher education, and many students in madrasahs and pesantren already know the rites in detail from manasik instruction. A mathematical analysis of tawaf and sa'i could therefore provide a context that is both familiar to these students and mathematically substantial.

Realizing this potential, however, raises a methodological problem. Ethnomathematics has been criticized for projecting school mathematics onto cultural practices, so that researchers find "circles" and "lines" that reflect their own categories rather than the logic of the practice (Barton, 1996; Vithal & Skovsmose, 1997). Descriptions of tawaf as "a circle" and sa'i as "a line segment" illustrate this risk: they are not wrong, but they say little that is specific to the rites and nothing that a mathematics teacher could not have said without studying them. A rigorous analysis must therefore do two things at once: it must build mathematical models that yield non-trivial results, and it must ground each model in the insiders' own description of the practice.

Two frameworks together meet this requirement. Ethnomodelling distinguishes the emic perspective of practitioners, the etic perspective of academic mathematics, and a dialogical perspective that relates the two without letting one absorb the other (Rosa & Orey, 2013). The Extended Theory of Mathematical Connections (ETMC) supplies a fine-grained vocabulary for the links established in that dialogue, distinguishing intra-mathematical, extra-mathematical, and ethnomathematical connections and several subtypes of each (Rodríguez-Nieto et al., 2021). ETMC has been applied to classroom teaching, students' problem solving, and local cultural practices such as cooking and brick-making (Rodríguez-Nieto & Escobar-Ramírez, 2022; Rodríguez-Nieto et al., 2025), and more recently to neuro-mathematical connections (Cantillo-Rudas, 2025; Cantillo-Rudas et al., 2024), but it has not yet been used to analyze collective ritual movement. Combining the two frameworks allows each mathematical result to be traced back to a specific emic rule and classified by the type of connection it represents.

Building on this combined framework, this study aims to model the spatial-geometric and kinematic structure of tawaf and sa'i and to examine how this structure connects ritual knowledge with school and university mathematics. It addresses three research questions: (1) What spatial-geometric and kinematic structures are embedded in the prescribed movement of tawaf and sa'i, as described in authoritative ritual sources? (2) How can these structures be modelled mathematically, and what results follow from the models? (3) How can the emic descriptions and the etic models be connected through the ETMC categories, and what learning trajectory does this suggest for geometry education? In answering these questions, the study seeks to extend ethnomathematics from static artifacts to collective ritual movement and to offer a replicable procedure for grounding mathematical models in emic rules.

2. Theoretical Framework

2.1 Ethnomathematics and Ethnomodelling

D'Ambrosio (1985, 2006) defined ethnomathematics as the mathematics practiced by identifiable cultural groups. Bishop (1988a, 1988b) identified six universal mathematical activities found in nearly all cultures: counting, locating, measuring, designing, playing, and explaining. Several of these activities are directly visible in tawaf and sa'i: pilgrims count circuits and passages, locate themselves with respect to fixed reference points, and follow a designed path. Rosa and Orey (2011, 2016) argued that ethnomathematics is not only a description of cultural objects but also a pedagogical action that bridges academic mathematics and the mathematical knowledge that emerges from everyday practice.

Ethnomodelling refines this position by distinguishing three perspectives (Rosa & Orey, 2013). The emic perspective describes a practice in the terms used by its members, the etic perspective describes it in the terms of an external system such as academic mathematics, and the dialogical perspective examines how the two can be brought together without one absorbing the other. This distinction is important because critics have warned that ethnomathematics can turn into "the mathematics we see in their practice," in which researchers simply project school concepts onto cultural objects (Barton, 1996; Vithal & Skovsmose, 1997). In this study, every etic model is therefore anchored in an explicit emic rule taken from authoritative ritual sources.

2.2 Extended Theory of Mathematical Connections

Mathematical understanding is built by connecting concepts, procedures, and representations (National Council of Teachers of Mathematics [NCTM], 2000). Businkas (2008) described mathematical connections as relationships between two or more mathematical ideas, and Eli et al. (2011, 2013) studied the connections of prospective teachers in geometry. Rodríguez-Nieto et al. (2021) extended this work into ETMC by networking it with the onto-semiotic approach. ETMC distinguishes intra-mathematical connections, which relate mathematical objects to one another, from extra-mathematical connections, which relate mathematics to other disciplines or to everyday contexts, and from ethnomathematical connections, which relate mathematics to the knowledge and practices of a cultural group (Rodríguez-Nieto & Escobar-Ramírez, 2022). More recent work has added neuro-mathematical connections (Cantillo-Rudas, 2025; Cantillo-Rudas et al., 2024).

Intra-mathematical connections are further divided into types. Table 1 summarizes the types used as the codebook in this study, adapted from Businkas (2008), Eli et al. (2011), and Rodríguez-Nieto et al. (2021). The richness of these categories makes ETMC suitable for analyzing a practice that may contain several types of connection at the same time, as tawaf and sa'i do.

Table 1

Codebook of Mathematical Connection Types Used in the Analysis

Connection type	Definition	Example indicator in the data
Feature	A concept is identified by its defining properties	The Ka'bah is a fixed reference that all points of the path surround
Different representations	The same object is expressed in alternate or equivalent registers	A verbal rule ("seven circuits") expressed as an angle of $7 \times 2\pi$ or as a graph
Procedural	A rule or algorithm is used to obtain a result	Computing the length of a circuit at a given distance from the Ka'bah
Implication	One property logically follows from another	An odd number of passages implies that sa'i ends at Marwah
Part-whole	A relation of inclusion or generalization	One circuit as a part of the complete tawaf; a circle as a special offset curve
Reversibility	A process can be undone or traversed in both directions	Sa'i passages alternate direction; tawaf has one fixed orientation
Meaning	The sense attributed to a concept in context	"One shawṭ" means a closed loop in tawaf but a one-way passage in sa'i
Modelling (extra-mathematical)	A mathematical model describes a non-mathematical situation	Distance-time graphs of raml and harwalah
Ethnomathematical	Mathematics is linked to the knowledge and practice of a cultural group	The counting unit shawṭ and the rule of keeping the Ka'bah on the left

2.3 The Geometry of Paths, Embodied Experience, and Geometric Thinking

A path in the plane can be closed, returning to its starting point, or open, with two distinct endpoints. A simple closed curve divides the plane into an inside and an outside, and a closed path drawn around a point has an orientation and a winding number that counts how many times it turns around the point (Needham, 1997). When a closed path keeps a fixed distance r from a convex region K , it is the boundary of the parallel body of K , and Steiner's formula states that its length equals the perimeter of K plus $2\pi r$ (Schneider, 2014). These results belong to secondary and university mathematics, but

their intuitive content (going around, coming back, moving farther out) is accessible much earlier.

The van Hiele model describes geometric thinking as moving from visualization through analysis and informal deduction to formal deduction and rigor (Burger & Shaughnessy, 1986; Crowley, 1987; van Hiele, 1999). At the visualization level, shapes are recognized holistically; at the analysis level, they are recognized by their properties (Fuys et al., 1988). Clements and Battista (1992) emphasized that spatial reasoning develops through direct experience of shapes and space, and research on embodied cognition argues that many mathematical ideas, including paths, containers, and motion, are grounded in bodily experience (Lakoff & Núñez, 2000). Tawaf and sa'i are performed with the whole body over an extended period, which makes them a natural source of embodied experience of closed and open paths, direction, and rate.

3. Method

3.1 Research Design

This study used a qualitative design combining document analysis (Bowen, 2009) and visual content analysis (Krippendorff, 2018) within an ethnomodelling framework (Rosa & Orey, 2013). A document-based design was chosen because research intervention during worship is neither ethical nor practical; pilgrims cannot be interrupted or interviewed while performing the rites. The analysis was organized in three phases that correspond to the emic, etic, and dialogical perspectives: (1) emic description of the prescribed structure of the rites, (2) etic geometric and kinematic modelling of that structure, and (3) dialogical coding of the links between the two using the ETMC codebook (Table 1).

3.2 Data Sources

The unit of analysis was not the individual pilgrim but textual units (sentences or paragraphs describing how the rites are performed) and visual units (scenes or frames showing movement patterns). Data sources were selected purposively (Etikan et al., 2016) according to three criteria: (a) the source explicitly describes the position, direction, path, or counting of tawaf or sa'i; (b) the source is issued by an official body or has undergone editorial review; and (c) visual sources show movement patterns without identifying individual pilgrims. Sources that dealt only with supplications or legal rulings without spatial information were used as context, not as primary units of analysis. Table 2 summarizes the corpus.

Table 2

Corpus of Data Sources

Code	Source	Type	Units coded
D1	Manasik guidebook issued by the Ministry of Religious Affairs of the Republic of Indonesia, covering the prescribed movement and counting rules of tawaf and sa'i.	Official guidebook	Textual descriptions of ritual rules; unit count not reported
D2	Umrah guide issued by the Saudi Ministry of Hajj and Umrah, describing the sequence, direction, and completion of the rites.	Official guidebook	Textual descriptions of ritual rules; unit count not reported
D3	Fiqh text describing the prescribed rules of tawaf and sa'i, used to interpret the normative spatial and counting requirements.	Scholarly text	Textual descriptions of ritual rules; unit count not reported
V series	Publicly available official visual records showing tawaf and sa'i from aerial and ground perspectives.	Video and photographs	Scenes and frames showing movement patterns; unit count not reported
L series	Academic literature on Islamic geometry, ethnomathematics, and ETMC	Articles and books	Used for triangulation

3.3 Analytical Procedure

In the emic phase, textual and visual units were reduced to a set of prescriptive rules, recorded wherever possible in the terms used by the sources (for example, shawṭ for one counted unit of movement, ar-rukn for a corner of the Ka'bah, raml and harwalah for the prescribed changes of pace). Each rule was retained only if it was stated in at least two independent sources. In the etic phase, each rule was translated into a mathematical statement, and the statements were combined into geometric and kinematic models. Idealizing assumptions were stated explicitly, and every derived result was checked for its dependence on these assumptions. In the dialogical phase,

each link between an emic rule and an etic statement was coded with the ETMC codebook (Table 1). The data were reduced, displayed, and interpreted following Miles et al. (2014), and conclusions were drawn by comparing the structures of the two rites.

3.4 Trustworthiness

Credibility was supported by source triangulation across guidebooks, scholarly texts, visual records, and academic literature (Lincoln & Guba, 1985; Patton, 2002). Two researchers coded the units independently; no numerical inter-coder agreement coefficient is reported. Dependability was supported by an audit trail linking the results to the emic rules, source codes, and modelling assumptions. Confirmability was supported by checking the etic interpretations against the normative descriptions of the rites. The dialogical analysis refers to the analytical comparison of emic and etic descriptions rather than validation through participant dialogue or documented external expert review. Regarding positionality, the first author lives and works in Saudi Arabia, and the second and third authors are Indonesian mathematics educators; this combination provided familiarity with the rites alongside mathematical education perspectives.

3.5 Ethical Considerations

The study did not involve human participants, and all visual materials were publicly available and did not identify individuals. The geometric description is analytical and structural; it does not evaluate the religious meaning or validity of the rites. Following D'Ambrosio (2006), the cultural-religious practice is treated as a source of implicit mathematical knowledge without reducing or reinterpreting its spiritual meaning.

4. Results and Discussion

4.1 Results

4.1.1 Emic Structure of the Two Rites

The emic phase produced a set of prescriptive rules that were consistently stated across the sources. Table 3 lists these rules, their sources, and the etic mathematical statement into which each was translated. The rules concern the reference points of the movement, the direction of movement, the counting unit, the region that must be included, and the prescribed changes of pace.

Table 3

Emic Rules of Tawaf and Sa'i and Their Etic Interpretation

Rite	Emic rule (as stated in the sources)	Sources	Etic mathematical statement
Tawaf	Begin and end each circuit at the corner of the Black Stone (ar-rukn al-aswad)	D1–D3; V series	A fixed ray from the center through the Black Stone corner serves as the counting line
Tawaf	Keep the Ka'bah on the left side	D1–D3	The path is positively oriented (counterclockwise seen from above)
Tawaf	Circle outside Ḥijr Ismā'īl, which is regarded as part of the Ka'bah	D1, D3	The obstacle is $K = \text{Ka'bah} \cup \text{Ḥijr}$, modelled by its convex hull
Tawaf	Complete seven circuits (ashwāt)	D1–D3	Winding number $w = 7$; accumulated angle 14π
Tawaf	Men walk briskly with short steps (raml) in the first three circuits of the arrival tawaf	D1, D3	Piecewise-constant angular speed, higher on circuits 1–3
Tawaf	Tawaf may be performed on the ground court or upper levels of the mosque	D2; V series	A family of nested closed paths at different distances r and heights
Sa'i	Begin at Ṣafā and end at Marwah	D1–D3	An open path with two distinct endpoints
Sa'i	Ṣafā to Marwah counts as one shawṭ, and the return as another	D1–D3	The counting unit is a one-way passage (a half period)
Sa'i	Complete seven ashwāt	D1–D3	$n = 7$ passages; position after n passages depends on the parity of n
Sa'i	Men jog (harwalah) between the two green markers	D1, D3; V series	Piecewise-constant speed with a faster middle interval

One emic feature deserves particular attention. The sources use the same word, shawṭ (plural ashwāt), as the counting unit for both rites. In tawaf, one shawṭ is a complete closed loop that returns to the Black Stone corner; in sa'i, one shawṭ is a single one-way passage. The same word and the same number seven therefore refer to two

different mathematical units, a full period in tawaf and a half period in sa'i. This difference is the starting point of the structural contrast developed below.

4.1.2 Tawaf: A Closed Curve Around a Convex Obstacle

Earlier descriptions of tawaf as "a circle around the Ka'bah" are imprecise. The Ka'bah is approximately a cuboid, and the path must also go around Ḥijr Ismā'īl, the low semicircular wall adjoining one side of the building. The obstacle that the path surrounds is therefore a non-circular region K . We model K by the convex hull of a rectangle with sides a and b and a half-disc attached to side b (Figure 1a). This idealization is used only for illustration; the main results below hold for any convex obstacle. A pilgrim who stays at a distance r from K walks along the boundary of the parallel body of K :

$$K_r = K \oplus D(r) = \{x \in \mathbb{R}^2 : \text{dist}(x, K) \leq r\}, C_r = \partial K_r \quad (1)$$

Steiner's formula for convex regions (Schneider, 2014) gives the length of one circuit and the area enclosed by it:

$$L(r) = P_K + 2\pi r, A(r) = A_K + P_K r + \pi r^2 \quad (2)$$

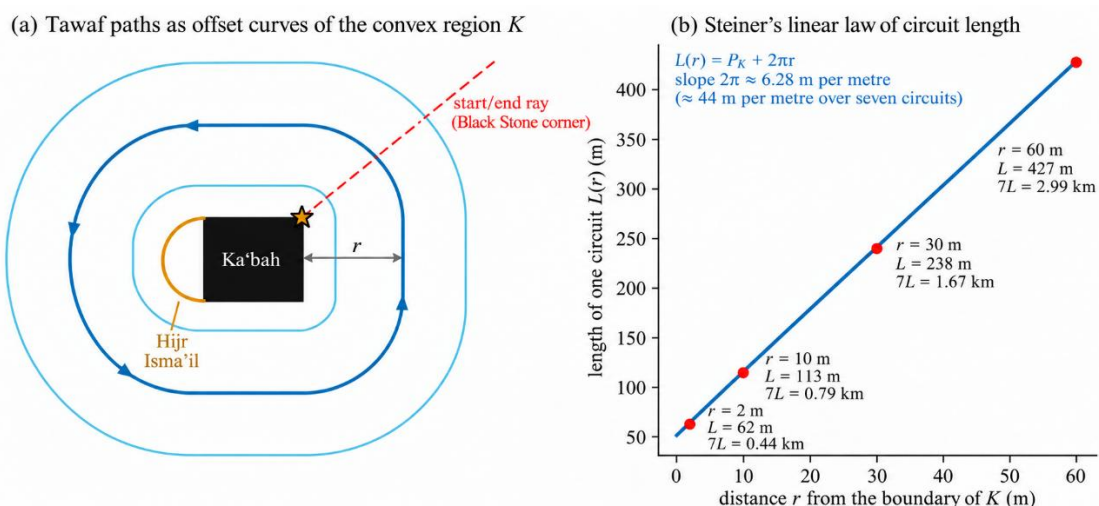
where P_K and A_K are the perimeter and area of K . Equation (2) shows that the length of a circuit increases linearly with the distance from the Ka'bah, with slope 2π , regardless of the exact shape and size of the building. Each additional metre of distance adds about 6.28 m to one circuit and about 44 m to the complete tawaf of seven circuits:

$$S_T(r) = 7L(r) = 7P_K + 14\pi r \quad (3)$$

For an illustrative idealization with $a = 12$ m, $b = 10$ m, and a half-disc of radius $b/2$, $P_K \approx 49.7$ m. A circuit then measures about 62 m at $r = 2$ m, 113 m at $r = 10$ m, 238 m at $r = 30$ m, and 427 m at $r = 60$ m, so the complete tawaf ranges from about 0.44 km close to the Ka'bah to about 3 km in an outer ring (Figure 1b). The exact numbers depend on the idealization, but the linear law and its slope do not. The offset curves are also smoother than K : their corners are replaced by circular arcs of radius r , so that the farther a pilgrim is from the Ka'bah, the more nearly circular the path becomes. This explains why tawaf is perceived as circular even though its center is not a point.

Figure 1

Tawaf as a Family of Offset Curves Around the Convex Region K (a) and the Linear Growth of Circuit Length with Distance (b)



Note. The dimensions of K are an illustrative idealization and are not to scale; the linear law $L(r) = P_K + 2\pi r$ holds for any convex K .

The rule of keeping the Ka'bah on the left fixes the orientation of the path. Seen from above, the pilgrim moves counterclockwise, which is the positive orientation in mathematics. The rule of beginning and ending each circuit at the Black Stone corner defines a ray from the interior of K through that corner, and a circuit is counted each time the pilgrim crosses this ray. Counting signed crossings of a ray is precisely how the winding number of a closed curve around a point O inside K is computed (Needham, 1997):

$$w(\gamma, O) = (1/2\pi) \oint \phi_\gamma d\theta = 7, \Theta_{\text{total}} = 14\pi \quad (4)$$

The completed tawaf is therefore a closed path with winding number seven, and its net displacement is zero, because the pilgrim returns to the starting point. Two further properties follow. First, because the path must stay outside K , it cannot be shrunk to a point without crossing the Ka'bah; topologically, it is a loop in the plane with K removed, and its winding number cannot be changed by continuous deformation. Second, the direction is not reversible: walking clockwise would produce winding number -7 , which does not satisfy the rule. Finally, because tawaf may be performed on the ground court or on upper levels, the set of possible paths forms a family of nested closed curves at different heights around the same vertical axis.

4.1.3 Sa'i: An Open Segment, Parity, and Displacement

Sa'i has a different structure. Its reference points are two endpoints, Şafā ($x = 0$) and Marwah ($x = L$), connected by a corridor that is straight in its idealized form. The emic rule that Şafā to Marwah counts as one shawṭ and the return as another makes

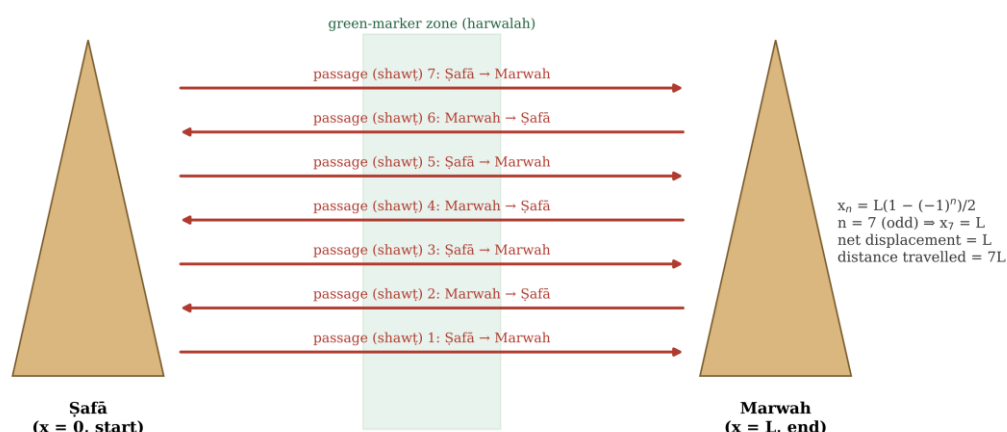
the counting unit a one-way passage. Let x_n denote the position of the pilgrim after n passages. Then

$$x_n = L(1 - (-1)^n)/2, \text{ so } x_n = L \text{ for odd } n \text{ and } x_n = 0 \text{ for even } n \quad (5)$$

Because seven is odd, Equation (5) implies that sa'i must end at Marwah, which is exactly the emic rule. The rule "begin at Şafā, end at Marwah" is therefore not an additional convention but a logical consequence of the counting unit and the number seven (Figure 2). The total distance walked is $7L$, while the net displacement is L . If L is taken as approximately 450 m (an illustrative model value rather than a measured corridor length), sa'i covers about 3.15 km but moves the pilgrim only one corridor length from the starting point. Sa'i thus provides a clear context for distinguishing distance (a scalar) from displacement (a vector), a distinction that in tawaf takes the extreme form of a positive distance with zero displacement.

Figure 2

Sa'i as Seven One-Way Passages on an Open Segment, With the Parity of the Final Position



In contrast to tawaf, the direction of movement in sa'i is reversible: each passage is followed by a passage in the opposite direction. As a function of time, the position of a pilgrim is a periodic, triangle-shaped function with period equal to two passages (Figure 3b). Seven passages correspond to three and a half periods, whereas seven circuits of tawaf correspond to seven full periods. The same emic count, seven ashwāt, thus encodes two different mathematical quantities.

4.1.4 Comparative Structure of the Two Rites

Table 4 brings together the structural properties of the two rites identified in the emic and etic phases.

Table 4

Structural Comparison of Tawaf and Sa'i

Property	Tawaf	Sa'i
Reference	One obstacle K (Ka'bah and Hijr Ismā'il)	Two endpoints (Şafā and Marwah)
Path type	Closed curve; offset curve of a convex region	Open path; line segment (idealized)
Orientation	Fixed, counterclockwise (positive); not reversible	Alternating; reversible
Counting unit (shawṭ)	One full loop (one period)	One one-way passage (half a period)
Seven units equal	Seven periods; winding number 7	Three and a half periods; ends at Marwah
Length of the rite	$7(P_K + 2\pi r)$, depends linearly on r	$7L$, independent of the pilgrim's position across the corridor
Net displacement	0	L
Change of pace	Raml in circuits 1–3 (men)	Harwalah between the green markers (men)
Vertical structure	Nested closed paths on several levels	Parallel corridors on several levels

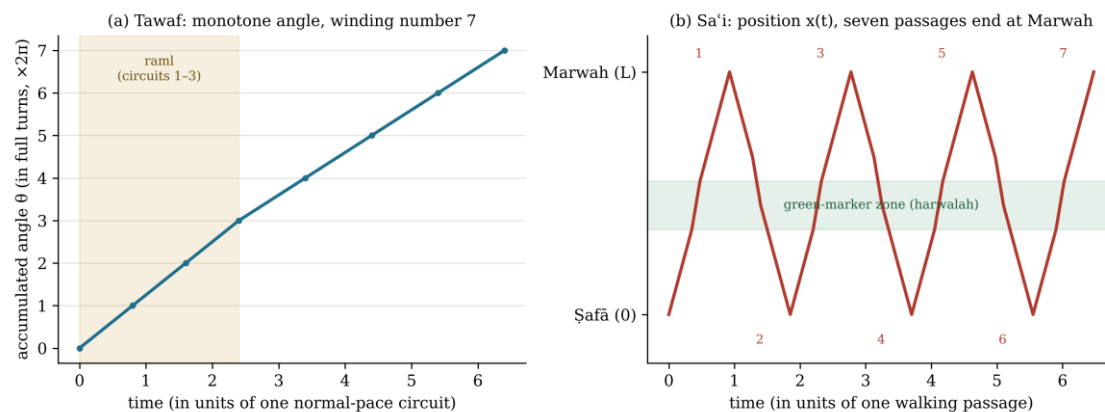
4.1.5 Kinematic Structure: Raml and Harwalah

The prescribed changes of pace add a kinematic layer to the geometric structure. In tawaf, the accumulated angle $\Theta(t)$ increases monotonically, and raml makes it increase faster during the first three circuits, so that $\Theta(t)$ is piecewise linear with a steeper slope at the start (Figure 3a). In sa'i, the position $x(t)$ is a triangle-shaped function, and harwalah makes it steeper within the green-marker zone (Figure 3b). If the green-marker zone has length ℓ_g and the walking and jogging speeds are v_w and v_j , the time needed for sa'i is

$$T_s = 7[(L - \ell_g)/v_w + \ell_g/v_j] \quad (6)$$

Figure 3

Kinematic Models: Accumulated Angle in Tawaf With Raml (a) and Position in Sa'i With Harwalah (b)



Note. Speeds are illustrative; the shape of the graphs, not the numerical values, follows from the emic rules.

The models also lead to a genuine optimization problem. Close to the Ka'bah, the path is short but the crowd is dense and walking speed is low; farther away, the path is longer but walking is faster. If $v(r)$ denotes the typical walking speed at distance r , the time needed for the complete tawaf is

$$T_T(r) = 7(P_K + 2\pi r) / v(r), \text{ with } dT_T/dr = 0 \Leftrightarrow 2\pi v(r) = (P_K + 2\pi r) v'(r) \quad (7)$$

Equation (7) shows that moving outward saves time only where the relative gain in speed, $v'(r)/v(r)$, exceeds the relative growth in path length, $2\pi/(P_K + 2\pi r)$. The model is not intended to advise pilgrims, but it illustrates how the rites can support authentic modelling tasks that involve rates, functions, and optimization.

4.1.6 Mapping the Findings Onto ETMC

In the dialogical phase, each link between an emic rule and an etic statement was coded with the ETMC codebook. Table 5 summarizes the connections identified and the evidence for each. All three main categories of ETMC were present in both rites, and all seven intra-mathematical connection types in the codebook were identified. Figure 4 summarizes how the emic, etic, and dialogical layers of the analysis are linked.

Table 5

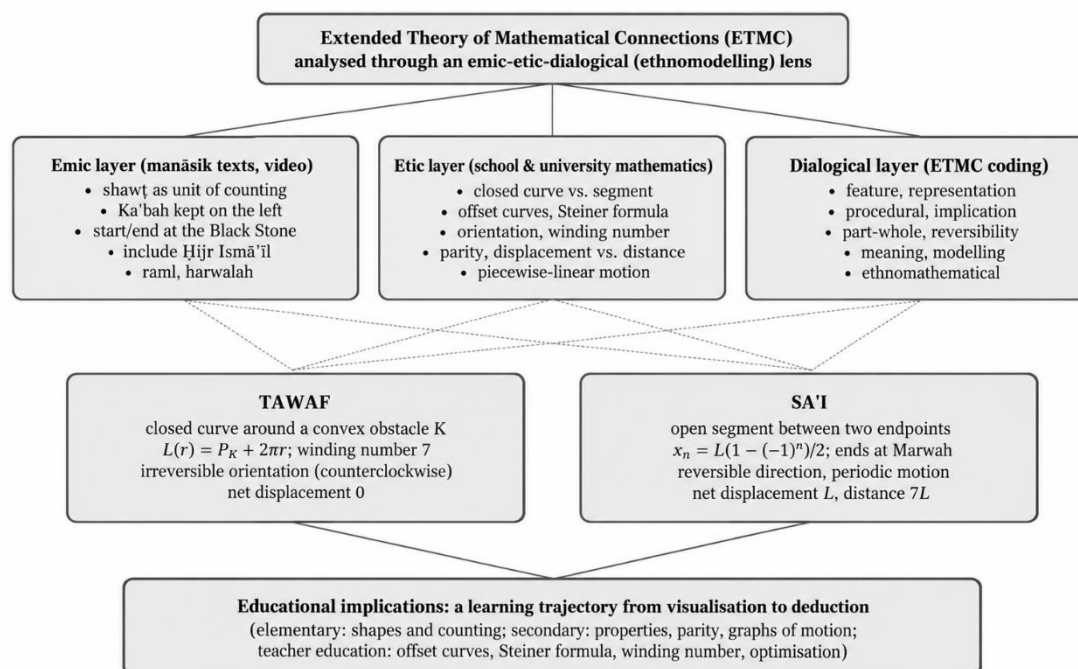
ETMC Connections Identified in Tawaf and Sa'i

Connection type	Tawaf	Sa'i
Feature	Fixed obstacle K; equal distance to K along each offset curve	Two fixed endpoints; shortest path between them
Different representations	Verbal rule $\leftrightarrow$ winding number $\leftrightarrow$ angle $14\pi \leftrightarrow$ graph of $\theta(t)$	Verbal rule $\leftrightarrow$ formula for $x_n \leftrightarrow$ triangle-shaped graph $x(t)$

Connection type	Tawaf	Sa'i
Procedural	Computing $L(r)$ and $S_T(r)$ with Steiner's formula	Computing the total distance $7L$ and the time T_S
Implication	Starting and ending at the same ray $\Rightarrow$ closed path, zero displacement	Seven is odd $\Rightarrow$ sa'i ends at Marwah
Part-whole	One circuit within the complete tawaf; the circle as the limiting shape of offset curves	One passage within the complete sa'i; one period equals two passages
Reversibility	Not reversible: clockwise motion gives $w = -7$	Reversible: each passage is followed by its inverse
Meaning	Shawt as a closed loop	Shawt as a one-way passage
Modelling (extra-mathematical)	$\Theta(t)$ with raml; optimization of $T_T(r)$; multi-level paths	$x(t)$ with harwalah; distance versus displacement
Ethnomathematical	Counting circuits from the Black Stone corner; keeping the Ka'bah on the left	Counting passages from Şafā; the green markers as boundaries of a speed interval

Figure 4

Framework Linking the Emic, Etic, and Dialogical Layers of the Analysis



4.1.7 From Analysis to Instruction: A Learning Trajectory

The results suggest a sequence of learning activities that moves through the van Hiele levels and across school stages. Table 6 presents this hypothetical learning trajectory (Simon, 1995). It is intended as a design proposal to be tested in future research, not as an empirically validated sequence.

Table 6

Hypothetical Learning Trajectory Based on Tawaf and Sa'i

Level (van Hiele)	School stage	Task based on the rites	Target concepts
Visualization	Elementary	Draw the path of tawaf and of sa'i from a video; decide which path "comes back" and which "goes and returns"	Closed and open paths; counting
Analysis	Upper elementary/junior secondary	Explain why sa'i ends at Marwah after seven passages; what would happen after six or eight?	Parity; endpoints; one-way unit
Informal deduction	Junior/senior secondary	Compare the distance walked and the displacement in each rite; sketch $\theta(t)$ and $x(t)$ with raml and harwalah	Distance and displacement; graphs of motion; rate
Formal deduction	Senior secondary	Show that every additional metre from a rectangle adds 2π m to a surrounding path (corners become quarter circles)	Perimeter of offset curves; π
Rigor	Teacher education/university	Prove Steiner's formula for convex polygons; interpret tawaf as a loop with winding number 7; analyze the optimization in Equation (7)	Parallel bodies; winding number; optimization

4.2 Discussion

This study set out to determine whether the prescribed movement of tawaf and sa'i contains mathematical structure beyond the familiar labels of "circle" and "line segment," and whether that structure can be grounded in the rites' own rules rather than imposed on them. The results answer both questions affirmatively. The discussion below interprets the four principal findings in relation to previous research, considers alternative interpretations, and draws out their theoretical and pedagogical implications.

4.2.1 From Naming Shapes to Deriving Results

The first finding is that modelling the rites produces mathematical results that naming shapes cannot. Describing tawaf as a circle yields no further statement, whereas describing it as an offset curve around a convex obstacle yields Steiner's linear law, the explanation of why the path looks increasingly circular with distance, and the winding number as a formal account of "seven circuits." Likewise, describing sa'i as a segment says nothing about where it ends, whereas the passage-counting model shows that the endpoint is fixed by parity. This distinction matters because a recurring weakness of ethnomathematical studies is that analyses stop at identifying shapes that students already know (cf. Barton, 1996). Freudenthal (1983) argued that mathematical structures are didactically valuable when they organize phenomena, that is, when they explain something about the phenomenon that would otherwise remain unexplained. In this sense, the models developed here organize the rites: they explain why distance from the Ka'bah matters so much for the length of tawaf and why sa'i must end at Marwah.

An alternative interpretation is that these results reflect the analysts' mathematical sophistication rather than anything in the rites, since Steiner's formula and winding numbers are not part of the pilgrims' explicit knowledge. This objection is important, and it is the reason why each model was anchored in an emic rule. The rites do not contain the formula, but they contain the conditions from which the formula follows: a fixed obstacle that must be kept on one side, a path that keeps its distance from it, and a count that is incremented at a fixed line. The mathematical results are therefore implicit in the organization of the practice, even if they are not explicit in the practitioners' discourse. This is the same position Gerdes (1986) took when he argued that the mathematics "frozen" in artifacts can be recovered by reconstructing the constraints under which they were made. The difference is that in tawaf and sa'i the mathematics is not frozen but enacted each time the rites are performed.

4.2.2 Grounding Etic Models in Emic Rules

The second finding concerns the relationship between the emic and etic perspectives. In two cases, the analysis showed that an emic rule and an etic result coincide. The rule that sa'i begins at Şafā and ends at Marwah is stated in the sources as a ritual requirement, and the model shows that it is also a logical consequence of the emic counting unit and the number seven. The rule that each circuit of tawaf begins and ends at the Black Stone corner is stated as a reference for counting, and it coincides with the standard method of computing a winding number by counting crossings of a ray. These coincidences are the strongest form of evidence that the mathematics identified here is grounded in the practice rather than projected onto it, and they respond directly to the criticisms of Barton (1996) and Vithal and Skovsmose (1997).

This result also refines the ethnomodelling framework. Rosa and Orey (2013) described the dialogical perspective as a negotiation between emic and etic

descriptions, but they did not specify how the quality of that negotiation could be judged. The present study suggests a practical criterion: a dialogical link is strong when an emic rule can be derived from, or shown to be equivalent to, an etic result, and weak when the etic description merely renames an emic feature. By this criterion, the parity of sa'i and the counting line of tawaf are strong links, the offset-curve model is a moderately strong link that depends on idealization, and the labels "circle" and "segment" are weak links. Researchers could use this gradation to evaluate the depth of ethnomathematical analyses in other contexts.

4.2.3 Meaning Connections and the Counting Unit

The third finding is that the shared emic term *shawṭ* carries two different mathematical meanings: a full period in tawaf and a half period in sa'i. In ETMC terms, this is a meaning connection, and it was the most productive connection in the analysis, because recognizing the difference in meaning is what makes the comparison of the two rites mathematically informative. Previous ETMC studies have identified meaning connections mainly in classroom discourse, for instance in how a teacher attributes meaning to the derivative (Rodríguez-Nieto et al., 2022, 2023) or how meanings structure classroom connections (De Gamboa et al., 2023). The present study shows that meaning connections can also be located in a cultural term, and that such a term can hide a mathematical distinction that becomes visible only when the practice is modelled. This extends ETMC from a framework for analyzing connections made by teachers and students to one that can analyze the mathematical knowledge embedded in a community's vocabulary, consistent with its onto-semiotic foundations (Font & Contreras, 2008; Rodríguez-Nieto et al., 2021).

The finding also has a direct pedagogical value. Students often treat counting as context-free, yet the question "what counts as one?" is fundamental to measurement, to periodic functions, and to modelling. The difference between one *shawṭ* in tawaf and one *shawṭ* in sa'i gives teachers a natural way to raise this question, and it connects counting, one of Bishop's (1988a) universal activities, with the idea of a unit in measurement and the period of a function.

4.2.4 From Shape to Motion

The fourth finding is that *raml* and *harwalah* add a kinematic structure that can be represented by piecewise-linear graphs and that the models support a genuine optimization problem. This extends the analysis from the geometry of the path to the motion along it, which is consistent with Clements and Battista's (1992) view that spatial reasoning includes the understanding of movement and transformation, and with embodied accounts in which motion along a path is a basic source of mathematical meaning (Lakoff & Núñez, 2000). It also addresses well-documented learning difficulties. Students frequently confuse position, distance, and velocity when reasoning about one-dimensional motion (Trowbridge & McDermott, 1980) and misread graphs of motion as pictures of the path (Leinhardt et al., 1990). Sa'i offers an

unusually clear contrast between the two notions, with a distance of $7L$ and a displacement of L , and tawaf offers the extreme case of a long distance with zero displacement. The graphs of $\theta(t)$ and $x(t)$ are also visibly different from the paths themselves, which can help students separate the shape of a graph from the shape of a motion.

4.2.5 Universality, Transfer, and Responsible Use

The rites of tawaf and sa'i differ from the regionally specific contexts that dominate ethnomathematics research, such as Kampung Kuta architecture, Yogyakarta batik, or Bedouin weaving (Amit & Abu Qouder, 2017; Prahmana & D'Ambrosio, 2020; Wijayanti et al., 2026). Their structure is identical across countries and cultures, which makes the context potentially relevant to Islamic mathematics education in many countries. Within realistic mathematics education, contexts are most effective when students can reason within them before formal mathematics is introduced, so that informal models can gradually become models for more general reasoning (Gravemeijer, 1999). Students who have learned the rites through manasik instruction already possess such informal knowledge, and the learning trajectory in Table 6 is designed to build on it.

At the same time, the use of a religious practice in mathematics instruction requires care. The geometric description is analytical and structural; it does not explain why the rites are performed, and it should not be presented as revealing a hidden purpose behind them (D'Ambrosio, 2006; Rosa & Orey, 2011). Teachers who use this context should present the mathematics as one way of describing the movement, respect the religious meaning of the rites, and avoid tasks that trivialize them. Used in this way, the context can show students that the mathematics they learn in school is also present in practices that are meaningful to them, which is one of the central aims of ethnomathematics as a pedagogical program.

4.3 Limitations

This study has several limitations. The source inventory does not report complete bibliographic details, visual-record timestamps, or coding-unit counts, which limits independent reconstruction of the corpus. First, it was based on documents and visual records and did not examine how pilgrims, manasik instructors, or students themselves perceive the geometric structure of the rites. Second, the geometric models rely on idealizations: the dimensions of the Ka'bah and Hajar Isma'il were simplified, the sa'i corridor was treated as a straight segment, and the speeds were illustrative. The structural results (the linear law with slope 2π , the winding number, the parity of the endpoint) do not depend on these idealizations, but the numerical values do. Third, the optimization model assumes a smooth relationship between distance and walking speed that would need to be estimated from crowd data. Fourth, the learning trajectory in Table 6 is hypothetical and has not yet been tested with students. Future research should interview pilgrims and manasik instructors, test the trajectory through design

research in madrasahs and pesantren, and use crowd-movement data to estimate the models empirically.

5. Conclusion

This study shows that the two pillars of umrah, tawaf and sa'i, embody contrasting and mathematically precise spatial structures. Tawaf is a positively oriented closed path with winding number seven around a non-circular convex obstacle, and the length of each circuit grows linearly with the distance from the Ka'bah according to Steiner's formula, $L(r) = P_K + 2\pi r$. Sa'i is a reversible open path whose endpoint is fixed by the parity of seven one-way passages, so that the ritual rule of ending at Marwah is also a mathematical consequence of the counting unit. The prescribed changes of pace, raml and harwalah, add a kinematic structure that can be represented by piecewise-linear graphs and supports modelling and optimization tasks. Using ETMC within an emic-etic-dialogical framework, the study identified intra-mathematical, extra-mathematical, and ethnomathematical connections in both rites and showed that the shared emic term shawṭ carries two different mathematical meanings. The study contributes to ethnomathematics by extending its scope to collective ritual movement and by offering a replicable procedure for grounding mathematical models in emic rules, and it contributes to mathematics education by proposing a learning trajectory that can be tested in Islamic educational settings.

Statements and Declarations

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Competing Interests:

The authors declare no conflict of interest.

Ethics Statement:

The study did not involve human participants. All visual materials analyzed were publicly available and did not identify individuals.

Data Availability:

The coded corpus and the audit trail supporting the findings are available from the corresponding author upon reasonable request.

Author Contributions:

Naif Mastoor Alsulami: Conceptualization, Investigation, Writing – original draft, Project administration. Sudirman: Methodology, Formal analysis, Writing – review & editing, Supervision. Muhamad Galang Isnawan: Data curation, Validation, Writing – review & editing.

Declaration of Generative AI Use:

ChatGPT (OpenAI) was used to assist with language editing and the refinement of draft

wording during manuscript revision. It was not used to generate data, analyses, or references. All AI-assisted suggestions were checked by the authors against the original sources and study records, and the authors retain full responsibility for the accuracy, interpretation, and integrity of the final manuscript.

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